Inception Games: Foundations and Concepts

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Abstract

The psychological science fiction movie, Inception, is based on synchronized group lucid dreaming, dream incubation, information extraction and idea injection. If a dreaming entity is not consciously aware that they are in a lower level reality, they can be influenced or coerced for sensitive information by more conscious individuals; something that would not have probably happened in original levels (reality). Deformations occur at each level so that spacetime compacted what-if epochs can be simulated as decision points along the history path of a conscious entity. In this study a toy conceptual model is proposed for inception coalition games. This metamodel is a novel abstract framework for generalized decision-making. Ideas emanating from automata theory, category/topos theory, physical causal models from quantum gravity, generalized theories of uncertainty, evolution, and spectral irrationality are used. Moreover, in generalizations to inceptions, game dynamics are proposed in which risk in strategies may be visualized through information morphing object interaction in multi-dimensional and sensorial virtuality. Conscious states born from different levels of inception and epistemic belief revision of strategies interact. Jumping to multiple levels will be equated with desiring information and influence peddling with time discounts. It is posited that inception games may be used as emergent risk analytics generated by recursive simulations of inception level games. Equilibria and pattern dynamics may be gleamed from these game constructs. The social impact of this study will be to present novel emergent approaches to decision-making that interact with general uncertainties and risk propagated by multiple hidden knowledge-seeking and effecting agents with diverse agendas.

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Introduction

The popular psychological science fiction movie, Inception, written, co-produced, and directed by Christopher Nolan, is based on group lucid dreaming, dream incubation, and coercive information extraction from individuals (Nolan, 2009). Levels of dreaming stages are shared by different people through the use of a fictitious apparatus, the PASIV (portable automated Somnacin intravenous) device that induces group sedation and synchronized dreaming. If a dreaming person is not consciously aware that they are in a lower level reality, they can be influenced or coerced for sensitive information by more conscious individuals; something that would not have probably happened in their level 0 (reality base) given their state of confidentiality and security surrounding such information.

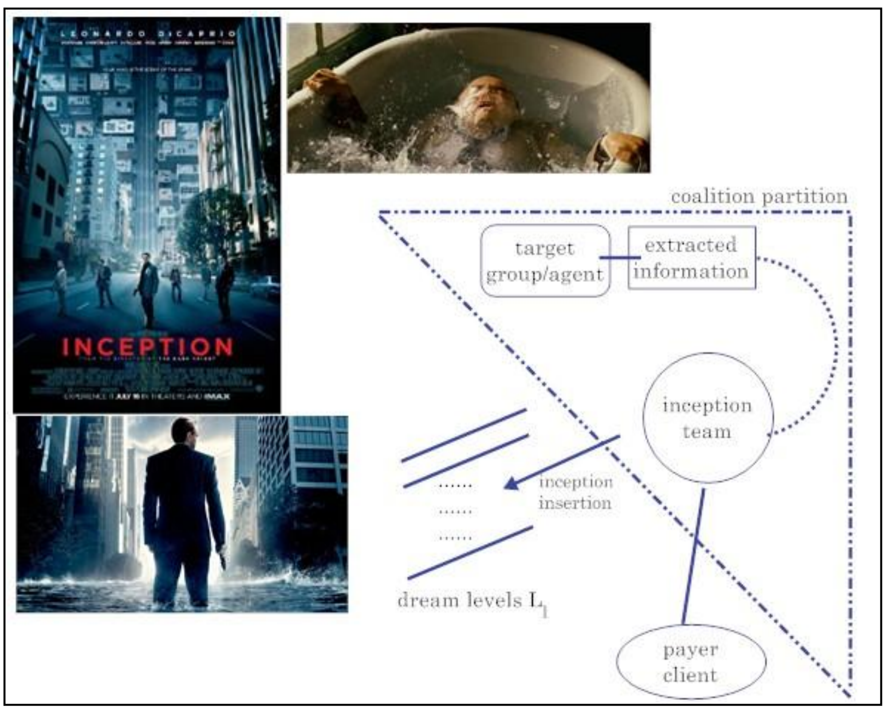

Figure 1 - The movie, Inception, its game concepts, gauging reality, and coalescing

In the übercompetitive environments of global business and geopolitics, all knowledge is acquired from the evolutionary and adaptive cyclical, iterative, and recursive sequences of observation, modeling, prediction, feedback, and coercion. This gestalt may be operated from combinations of experiential intuition, scientific exploration, and spectral irrationality (Slovic, Fischoff, & Lichtenstein, 1980). However, extraction of information is manipulative and resides on the fringes of espionage and coercion. In this way, inception-like games emulate real world information gathering, idea infusion, and coercion. It is in this spirit of computational gaming strategy that the author proposes to construct a novel decision structure that utilizes a generalization of inception-like rules and contemporary physical theories to emulate and model emergent co-opetive multi-agent organizational behavior dynamics involving coalitions.

Inception teams (those attempting the extraction of information from, thought injection into or influence pedaling of unaware individuals) are akin to subcoalition groups in noncooperative game theory (Debraj, 2007). One may then seek coalition strategy profiles that approach Nash, ε-Nash equilibria, evolutionarily stable, ε-evolutionarily stable, and other types of coherent strategy types in this subgenre of games that obtain and enforce evolvable inceptions or this study's definition of approximate inception, labeled ε-inceptions. Recent advances in generalizing probability for emergent physical theories such as digital (discrete) Planck level quantum-gravity and causaloid information structures, may lead to uncertainty game strategies that transcend both anthropomorphic and Planck-to-universe level spacetime mechanics (Hardy, 2008).

Additionally, generalizing game formats to explicitly express irrational social interaction on a spectrum and in the most general language of mathematics, category theory, and of theoretical quantum-like automaton, expands their implications for all physical and psychical action. Quantum strategies in quantum games have been posited to more adequately model bounded rationality by using quantum interference patterns, for example (Burns & Roszkowska, 2005; Vannucci, 2004). The expressiveness and generality of non-clasical mathematical logics bridged by category and automata theories and formed with respect to the structure of general languages ostensifies how powerful games as general multi-agent decision flow objects can describe behavior in sentient machines.

Inceptions may be more generally framed as (time) discounted recursive games with stochastic evolutionary dynamics. These uncertainty paradigms will be applied to inception game structures in order to produce novel emergent games that may mimic theoretical holistic co-opetive natural decision systems in the universe. In this study, inception branches are constructed as very general co-opetive dynamic stochastic games with belief revision dynamics involving social power as measurements of payoffs. Social power will be displayed in the form of *consciousness awareness* within each inception level. Here consciousness awareness will act as a surrogate measurement of psychological advantage of one agent or group over another in the struggle leading to inceptions.

These games are also manifested in a computational sense utilizing generalized evolutionary and recursive hypercomputation automaton in which time dilation, in general, and spacetime quantum gravity, in particular, are exploited by progenic generations of evolutionary automaton, in order to accelerate computation of the projected information coercion in each level.

General recursion plays a major role in the morphogenesis of inceptions. Antrhopomorphic thought and hence deiciosn-making can be viewed a structurely recursive, of a possible emergent fractal nature (Kurzweil, 2012). Additionally, as noted before, recursion is a powerful metaphor in automata and computation as a mechanism for general AI. Resursions are natural models for conflict – every thought process can be deconstructed as a type of conflict for inceptions. In this study inceptions may generalize conflict gaming and well as be the main motive for investigating belief revision dynamics.

These hypercomputers will be vested as Zeno-type machines. Recall that a Zeno machine (ZM) is a version of a hypercomputer in which infinitely many Turing machine (TM) operations can be computed in finite periods of time (i.e., exploiting the Zeno paradox of approaching a boundary in the interval [0,1] by fractionation) (Potgieter, 2006). These Zeno machines may then evolve into higher order Gödel-von Neumann machines that are capable of self-writing adaptive new code and self-generating new prodigy machines (Schmidhuber, 2006; von Neumann, 1966). Rather than rely on standard recursiveness, inceptions will then be expanded to describe holarchical strategy structures - self-similar nets of inclusive decision branches (graphs) that are manifested as repeated and connected holons in a pseudo hierarchical network that are toy models for universal natural organization (Koestler, 1967/1990; Wilber, 1996; Laszlo, 2004).

Belief revision is key to inceptions since one group is endeavoring to influence another's belief system so as to extract information or change that group's motives or actions. Belief revision in this sense, may be done through the use of manipulation of ideals which minimize contradictions within the inceptee's belief system and by utilizing general Bayesian causal belief nets that take into account emergent uncertainty models for human neuroeconomic behavior including general fuzzified states, payoffs, and strategies. Fuzzy logic and other uncertainty frameworks, including quantum probability and causaloids, are framed through the use of Zadeh's logical precisiation language setup known as general theory of uncertainty (GTU) constraints (Zadeh, 2006). Auxiliary to the GTU, the logical systems work of Gärdenfors (1992, 2004) and Dubois and Prade (2001) that extend Dempster-Shafer evidence theories of belief functions [Dempster (1968) and Shafer (1979)], and Zadeh possibility theory [Zadeh (1978)], and the social belief system modeling of Gilbert (1991), Hasson, Simmons, and Todorov (2005), and Glimcher (2010), help form what a mathematical description could be for such *belief equilibria* in inception games, the notion of a convergence to a consequence of belief revision and ensuing inception using inception (belief revision) operators as part of strategy profiles. Gabbay, Rodrigues, and Russo (2007) and others extend belief revision operators to non-classical first-order (predicate) logics which would include the emergent systems considered in our generalized representations using GTU constraints in this study. The category-theoretic representation of games given in Appendix E are mappable to higher order logics (*n*-categories and *n*-topoi have predicates which are ascending sets of sets and morphisms, etc.) and hence generalize belief revisions for first-order logics.

The use of virtual worlds (VWs) in simulating these decision games and visualizing risk will be instructive in building human-induced strategies under inception-like scenarios and their visualization. Simulations may also be run in order to observe inception game equilibria in human played VW game environments. However, an effective visualization of risk remains lacking in traditional or contemporary analytics. Just-in-time or better still, ahead-of-time analytics should be stimulated by illuminating and interactive analytics visuals such as generalized shapes that represent real-time and projective risk objects and the orthonormal relative frameworks for measuring those risk profiles. Traditionally risk is measured as an operator on the product space of utility and some uncertainty space of distributions, (i.e., the inner product of utility and uncertainty measure). What is usually not measured is the rationality of the uncertainty distributions made apriori and the propensity of the DM to be on a spectrum of risk-taking attitudes. These are invariably multi-dimensional, adaptive, and agenda (goal)-based from uncertain or incomplete information, hence the generality of the bounded rationality thesis of Simon (1957; 1991). Moreover, inceptions will be proposed as general super-recursive games and automata. Simulations of inception games will then take on the form of very general recursive simulations, (i.e., simulations within simulations acting as advantageous speedup parallizers for running decision event scenarios). In particular, recursive simulation of simple strategic games was found to dramatically produce improvement in tactical military gaming (Gilmer & Sullivan, 2000; Agarwal & Gilmer, 2004).

Organization of study

This paper will be organized into sections outlining the introduction, focus, objectives, importance, lack of study and gaps, limitations, concerns, literature review and past works pertaining to similar constructs, followed by the main discussion of definitions, building blocks, and approaches to inception-like games and strategies. The paper includes five appendices that detail the main constructs of (i) decision and game theory, (ii) emergent game theories connecting back to general stochastic recursive games which are being presented as frameworks for inceptions, (iii) causaloids – a quantum causality model for emergent probability in the context of a theory of quantum gravity, and to be utilized to generalize inceptions to physics-based emergent structures, (iv) an approach to a generalized expression of uncertainty used liberally, in this paper, to generalize the uncertainty regimes of games, decision, and information structures, and (v) category and topos theories which will lead to the development of higher order abstraction of general games and in particular, inception games. The introduction section is a summary of the premise surrounding inception, its rules, and mechanics. It will also encase a synopsis of the intent of the study to construct a mathematical representation of the inception-induced strategies into a game structure. Following this, as mentioned before, subsections on the focus and importance of the constructs, the lack of previous studies to investigate this phenomena, and objectives of the study will be presented. This section lays out a brief organizational guide to the paper.

The literature review will present with past piece-wise relevant components of game theoretic developments and emergent physical theories relevant to our approach.

Various concepts and proposals from coalition game theory, emergent decision and game theories and behavior economics, emergent physical information theories, category/topos and automata theories, and stochastic-recursive games will be presented. Following this, virtuality and visualization of uncertainty and risk in information and decisions/games and the psychology of lucid dreaming will be gleamed upon, contributing to the premise of how we will present inception games in interfaces.

In the next section, the methodologies used for the study will be discussed, including the interpretive and critical aspects of the methods. Following this will be a description of the study methodology and potential simulation set up of inception-like games and scenarios in virtuality interaction and computation.

The development and analysis of the emergent models proposed for inception games will follow in the next section, followed by a section on general discussions which will include a briefing on the study findings, relating them back to previous relevant and contributing studies.

In the next section, the limitations of the abstract models will be reviewed, including the shortfalls of any generalizations to the strategy models of the study with respect to real world scenarios. Following the section on limitations will be a section on the implications of the models developed for inceptions and their implications for social interaction analysis.

The paper will conclude with a synopsis of the study, its conclusions based on the models presented and the study's implications and impact on societal behavioral analysis. The paper will list all references and five appendices that introduce major aspects of the foundations of traditional and emergent decision/game theories, higher order category/topos generalizations for games, a theory of generalized uncertainty, and a causal structure for quantum-gravity as a general approach to physical-based probability that may be used on very general dynamic game constructs.

Focus of study

In this paper, the idea of abstracting patterns of behavior and stratagem of a highly dynamic and evolvable game operated on generalized inception-like rules of engagement in order to navigate in such environments may lead to a more comprehensive and technical understanding of coercive behavior in interacting organizations, large subgroups, cliques, and individual agents. The focus of this study is to propose a novel generalized game structure that evokes evolutional patterns of behavior and strategy choice in inception-like rule spaces where agents have agendas to gather information about or to use in transacting some interaction with other agents. This is the essence of a co-opetive market. These may lead to the development and formation of generalized agenda-based coalition subgroups in physically complex evolvable and adaptive environments.

Importance of study

In this study, insights into the operationalization and formation of working coalition groups in real world industries or government entities may be gleamed from the study of simulated inception games. This dynamic, in turn, may reveal some processes that lead to the morphogenesis of important coalitions that form to influence global economic and political phenomena. Inception-like activity introduces a natural version of coercion in transactions between groups and individuals. This may emulate modern, real world agenda-based techno-socio-economic behavioral patterns. Coercion is achieved more easily through the ubiquity of this global techno-socio-economic network and ensuing media types. Additionally, the time dilation properties of inception scenarios and the inconsistencies inherent in lucid dreaming in groups perpetuated by inceptions may point to temporal advantages in certain strategies and of the proposition that any inception level can be equated to a hyper-reality level which includes the observer's subjective reality level (level 0), (i.e., relativism in reality and its logic system inconsistencies ala Gödel's Incompleteness Theorem) (Gödel, 1962).

Lack of study

While the fields of behavior economics, game theory, and psychology have fused together quite remarkably in the last three decades, due largely to the seminal works of Tversky & Kahneman (1974), Gilovich, Griffin, & Kahneman (2002), and Finucane, Alhakami, Slovic, & Johnson (2000), obtaining profoundly technical insights into the workings of human decision-making under various conditions of uncertainty, spectrum of risk, and stress, no unifying framework encompassing emergent sciences nor the particular spacetime dilation branching and equated time discounting stochastic games described in this paper, has been studied or developed. Gilboa (2010) has pronounced that the field of decision theory remains lacking a uniformly cohesive calculus – different decision environments require different regimes of uncertainty, risk, and rationality measures. In addition, preference spaces are mostly approximated based on experimental outcomes – normative approaches to bounded rationality are prescriptive not theoretical. Again, a uniform approach to decision game spaces lacks a general framework. Inception-branching, as it will be labeled in this study, is akin to auto-generation of training sets in artificial neural networks (ANNs). However, there is a major difference. Training in ANNs tune the weights of the mostly static neural network structure, (i.e., the model order of the ANN remains essentially the same and the influences of agents are nearly homogenous). In inception-branching, coercion and persuasion may sway the participants – be they the intended target of inception or the inception team – to change the model of reality for that information regime.

Treating dream states as scenario builders, while previously posited (Revonsuo, 2000), has never been equated to or synthesized for decision analysis. Additionally, mathematical treatments of inception-like strategies for games have never been approached for utility or investigated to surmise possible types of game equilibria for projecting stability of stratagems. Inception is a different type of lucid dreaming because it considers multiple group interaction and hence is a type of group decision making with asymmetric weight influence distribution.

Research Questions

Some research questions that this study will attempt to confront include (i) can inception-like rules in a gamte theoretic setting emulate or improve real world decision making and strategy formation in organisms, (ii) are there intrinsic novelties in the game structure of inceptions or do they fall into the category of strategy models or patterns in game theory with certain useful and practical solutions/equilibria, (iii) can one frame belief revision (operators) on belief systems of coalitions for inceptions, (iv) using a virtual world/holodeck, can hybrid human-automaton multi-agent systems form emergently superior strategies not predictable using classi al rational decision/game theory, (v) can risk be a generalized multi-dimensional measure beyond intervals or singletons and can it be translated to the human as sensory stimuli, and (vi) are inception game structures representative of a more higher order abstract such as a logic, automata, or categories/topoi?

Objectives

In this study, the ultimate goal is to construct a viable model for inception-like games with strategies and propose their corresponding virtualization that employ very general mathematical rules emulating actions and scenarios in complex, adaptive, and evolutional organizational behavior. The game algorithms can be embedded in a virtual worlds (VW) environment such as Second Life (Gross, 2006) or a prototypical Star Trek style holodeck as the action engine. Human role playing can be monitored, to be compared to full simulations using game analytics (Medler and Magerko, 2011). The evolution of both experiments may then be compared based on typologies of patterns of action (strategy evolutional patterns) thereby ostensifying the behavior of agents and groups exhibiting inception-like agendas which can be viewed as general social interactions in organizations and organisms.

Additionally, abstract mathematical and contemporary physics-based theoretical and conceptual models of inception-like games and strategies framed and motivated in part from the psychological science fiction movie Inception will be presented. Virtuality of sensory systems mapped to risk components of inception games are posited. It is anticipated that such results can help develop future emergent game and decision analytics through the process of recursive game navigation in the multiple levels of inception scenarios. Games and automata are generalized in order to expand the concept of social inception games using emergent mathematical and physical theories.

General Theory, Concepts, and Hypotheses

In this study much will be made of the concepts of inception-like strategies in conflict games. It will be posited that inceptions are equivalent to general social stochastic (dynamic) recursive games with individual and coalitional strategies, behavioral rules, belief revision, general utility functions that describe diverse forms of payoff, and compacted time-discounted evolution. The mathematical models and proposed simulations in this study of inception-like behavior in virtual gaming is a means to emergent and evolutionarily strategic gaming scenarios producing advanced predictive risk analytics built from an equivalent environment of extended game analytics. Inception-like (dream) levels may be emulated in modeled virtual worlds to simulate emergence of strategic behavior in generalized conflicts involving coercive information flows. Through inception dynamics and novel representations thereof, generalized decision risk can be effectively and directly linked to human-machine sensorium, a kind of virtual enhanced hair on the back of your neck. Finally, inception games as highly dynamic social processes have more powerful abstractions and higer-level mathematical presentations as categories, topoi, logics, and automata.

Literature Review

In this section a review of classical and contemporary developments in coalition game theory, behavioral economics, emergent physical theories, non-classical approaches to automata and category theories, virtuality and holography for games, and the psychology of lucid dreaming will be briefly done. For brevity, the appendices contain broad overviews of classical and emergent non-classical decision and game theories including quantum, relativistic, fuzzy, and rough set, and causaloid-based quantum gravity as a novel proposal for a causal quantum gravity automata and physical theory.

Additionally, a generalized theory of uncertainty is reviewed in Appendix B as a framework to express general models for uncertainty. Under the umbrella of a generalized theory of uncertainty (GTU) from Zadeh (2006), most automata and corresponding game models will be generalized in expression. These fields represent the foundational cross section of the approach to the novelty of inception-like game strategies given in this study. In this treatise on inception-like levels in games, behavior economics is melted with game theory using the possible irrationalities involved in dream-like imaginative what-if scenarios with game strategies.

The psychology of lucid dreams is a starting point to allow for an analogy for stratagem thinking for inceptions. The bicameral nature of human decision making is central to the theme of multiply embedded inception levels. Gleamed from fNI (functional neuroimaging) studies, when making decisions humans rely on both the slow high level rational analysis emanating from the neural net circuitry of the prefrontal cortex (PFC) areas (ventromedial-vMPFC and dorsolateral-dlPFC) in conjunction with the orbitofrontal cortex (OFC), and the fast emotional and intuitive approaches arising from the neural net circuitry of combination of areas of the amygdala, fusiform gyrus, insulae cortex, among others (Kalin, Shelton, and Davidson, 2004; 2007; Singer, 2006; Glascher, Adolphs, Damasio, et al., 2012; D'Argembeau, 2013). Multiple inception levels may present with a concept that espouses various and different multiple combinations of components in the areas of the decision-making bicameral mind. For example, a certain inception level may concentrate on elements of risk aversion developing through the chilling fear of loss combined with the unemotional probabilistic analysis of updated Bayesian priors leading to posteriori distributions for those loses. Other inception levels may include different combinations of risk aggression and a different type of probabilistic analysis such as those emanating from quantum-based stochastics. The final inception may then be the master or controller of all these partial conjectures of decision-making from all inception levels developed.

While decision-making was just given as a human endeavor, here we shall approach a decision-making collective from a hybrid of entities, including automata. Emergent physical theories drive novel automata models which will give rise to a way to equate winning game strategies in inceptions (such as ε-Nash equilibrium and ε-evolutionarily stable strategies) with modular recursive calls (from inception level subgames) in emergent automata types that may emulate inceptions.

Behavioral and neuroeconomics

Game theory, the multi-agent interactive extension of decision theory, is an analytic tool to help describe what may transpire in a game situation where strategies (action space profiles) are executed and resultant payoffs made, under deterministic, non-deterministic, or probabilistic rules of engagement. Decision rationality is assumed for classical game theory depending mostly on utilizing rational choice theory (Scott, 2000). However, humans become irrational or more aptly, boundedly rational for many reasons, including fear, bias, levels of uncertainty, vagueness or lack of information, lack of time to analyze, aggression, and complacency (Simon, 1991). These situations manifest themselves so often in transactions that unpredictable irrationality plays a pivotal part in modern game portfolios, from stock market swings to consumption trends to black swan and dragon king catastrophes (Sornette, 2009; Taleb, 2007). This scenario is what the field of behavioral economics attempts to embrace. Cognitive psychologists, led by Kahneman and Tversky, showed how humans made decisions under uncertainty and emotion (Tversky & Kahneman, 1974). This precipitated a monumental change in the way economists described human decision-making economic behavior. In the inception game, dream states define the silo of irrational possibilities that can be applied to situational interactions from an individual's current inception level. This means that behavioral economic tenets may be applied to inception games involving dream state levels, the ultimate human what-if analyzer of things that could be physically impossible events in reality (modulo level-0 inception).

In large systems of interacting agents engaged in strategies, games become complex very quickly. It was Axelrod (1997) who endeavored to model this complexity in how large multi-agent systems evolve, finding patterns of formation, detachment, and ultimate stasis or chaos. Large scale game simulations bore out the idea that complex interacting systems evolve into discernible categories of social behavior, especially when cooperation on a spectrum is manifested. Group behavioral economics is then a complex of multi-agent economic agents that offer strategies which may gyrate from cyclic stable behavior to regimes of transition to chaotic regions. The interacting levers of cooperation temperature for agents dictate these asymptotic patterns for the group. Inception games represent a type of recursive and evolutional complexity in such multi-agent networks based on the meandering temporal levels of inception. Behavioral game theory and economics are reviewed in more detail in Appendix B.

Game theory and coalition building

Coalitions are built based on shared interests in game situations among subgroups of agents. Within a coalition, it is assumed that a cooperative game ensues as this shared interest is held. However, between coalitions, it is assumed that a non-cooperative game develops (Debraj, 2007). There are nonetheless, situations in which different mixed strategies (probabilistic mixtures) among agents are executed within coalitions (being unpredictable when others predict so) despite the shared interests, developing into mixed co-opetive groups of coalitions, (i.e., some coalitions share partial interests and partial counter-interests) (Brandenburger & Nalebuff, 1996; Camerer, 2003, 118-150). An example of this game situation would be counter-intelligence, multiple spy levels, and double-diplomacy (i.e., double, triple … N-agency with adjoined and composite dynamic self-interest agendas).

In inception-like games, multiple dream levels introduce complexity into and opportunities for coalitions to form based on collective consciousness awareness and advantages within subgroups of dreamers. This paper will investigate possible formation patterns for coalitions which would be the only way for an individual agent to survive in an inception game. Appendix A reviews some of the mathematical setup for classical and non-classical games and decisions, definitions, and novel approaches in which we would like to generalize to our scenario in inception games. In our main discussion, we start with some definitions of inception interaction, agents in groups or individually are measured for advantages based on their knowledge of what level they are in (consciousness level) and the simultaneous lack of consciousness of their targeted counterparts. Inceptions are then treated as generalized stochastic recursive games in which a rich equivalence between game graph strategies and pushdown automata modular calls is utilized to posit computationally favorable stratagem in inceptions.

Psychology and structure of dreams, visualization, and virtuality of games

Lucid dreaming (van Eeden, 1913) is the validated phenomena of perceiving dreams within dreams and of being conscious of being in a dream state (LaBerge, 1990; LaBerge, 2009). It is therefore a forerunner to the awareness of group dreaming iterations of inceptions. Group dreaming is not possible, but a group having similar dream scenarios is likely when that group experiences similar real world phenomena because of external stimuli and associative memories contributing to the staging of dreams. The physiology of entering into a dream state suggests that the mind solidifies its perception of incredible events in a dream by correlating them with real world phenomena that closely match them, (i.e., a freak event in a dream is linked with an equally unlikely, but similar realized event in the conscious world) (McLeester, 1976).

In inception dreaming, an individual (group) is required to be able to differentiate levels using several cues, such as the movie totem spin top spinning indefinitely in a dream state as opposed to eventually stopping in a reasonable amount of time in reality. However, an infinitely spinning top is possible in one's imagination or through hallucinations and hence in a dream state because one may have come to the conclusion in their reality (level 0) that some things are infinite in certain belief systems. This is an inconsistency in one's logic belief system. However, in any logic or (logical) belief system in which one can internally prove its own consistency, incompleteness of its proving power persists (Gödel, 1962).

Virtuality refers to attempts to asymptotically approximate a model of reality through exceedingly more interactive computer simulation of anthropomorphic sensing of stimuli and near real-time feedback in order to reconstruct a target reality. This compact description is not without ambiguity as Peirce defined virtuality in a metaphysical manner to mean that any reality is an approximation to connectives to objects – the thirdness of things (Peirce, 1931). The firstness of Peircean reality is the object's metaphysical self or essence. The secondness is the reflection of the observer of that self. Thirdness is usually approximated using signification – assigning expressions to objects. In this sense, the triad of Peircean reality create virtualities of the original object self (Esposito, 2013). Computational virtuality is optimally therefore a 4^{th} order approximation to reality, (i.e., computation, if universal, asymptotically converges to signification). Peeling back the onion of successively more computationally powerful automata reveals *nth* order approximations to Peircean signification based on SPACE, TIME, and SPACETIME computational resource orders of magnitude with respect to Planck-level granularity for digital physics reality.

Inceptions mimic socio-economic interactions – everyone wants something from someone. Virtualizing such game scenarios is akin to generating what-if computations for these interactions. If real-time computation of risk can be visualized while the dynamics of the decision and game branching proceeds in an inception, the structure of the metamorphosis of highly complex and adaptive games may be visually acknowledged. This enlightenment may make practical prescriptive and descriptive visual models for highly adaptive social dynamics. One purpose of this study is to propose risk virtual and visual tools that may convey and present more simultaneous information on real-time risks of decision-making within the structure of the generality of inception games as social interactive constructs. Most visualization queues used for quantitative interaction are militaristic metaphors – cockpit or dashboard digital dials, hybridized levers, digital meterizations, etc. Here we will consider more ethereal connectives to the sensorium of anthropoids – geometric shapes and generalized tactile/haptic interactions through multi-sensorial universes. Risk will be a multi-dimensional object whose orthogonal directions are independent measures of various aspects of uncertainty, irrationality, and social relational spaces.

Jiang (2011) develops temporal and Bayesian extensions to the idea of action graph games (AGGs) , in addition to deriving aspects of Nash equilibrium computational complexities. Basic AGGs (AGG-$\varnothing$) as introduced in Bhat and Leyton-Brown (2004), lead to a compact visualization of structured large games possessing simultaneous agent moves, independent action spaces and other simplifying conditions. Appendix A details what AGGs are, including the extensions from Jiang. Compact visualization of AGGs comes from a graph depicting nodes (agent actions), membership in neighborhoods for nodes (directed edges), and dotted containers (agent action spaces). Here, we may utilize aspects of AGGs extended to the more dynamic and evolutional properties of inception games, (i.e., to general multi-dimensional evolutionary games considered with emergent physical laws), and their visualization in multi-sensorium virtual environments as generalized *n*-dimensional and polymatrix-like displays in conjunction with the author's notion of compact visualization of game risk through the use of visual glyphs known as *i*-morphs. Modifications to AGGs for inception games present notions of network visualization while *i*-morphs aid in envisaging dynamic risk in inceptions.

In very simple 2x2 games (two agents with two moves), recent developments from Goforth and Robinson (2005), Robinson, Goforth, and Cargill (2007), and Bruns (2010;2011;2012) have led to a similarity categorization (and metric) of those games in terms of a graph topology with respect to payoff families and preference orderings. This is a graphical visualization of similar 2x2 games using swap moves that reshuffle preferences and payoff perturbations. Topological neighborhoods are defined by the number of swaps or moves that connect different similar games. Hence, a game topology for the finite discrete space of 2x2 games is a means to developing an effective similarity measure given by the number of swaps or moves needed to transform from one game to another. The merit of this type of typology is to visualize how games may be transformed into others, therein being able to more powerfully investigate the dynamics of social interactions and examining the possible resolution of conflict and destructive inequities. In this study we consider expanding this discrete topological view to higher games in an attempt to more readily visualize multi-dimensional risk in evolutionary games such as inceptions. The space of 2x2 games is easily categorized, but serves as atomic building blocks for more complex and general games, in particular games involving computational units that general the Boolean algebra such as our treatment of *qbits* or *e*-bits (entangled bits) in quantum games and further generalizations in our application of a generalized theory of uncertainty (GTU) from Zadeh (2006), manifesting a generalized bit labeled as a *g*-bit.

The geometry of games has played an extremely important and central role in the understanding and computation of equilibria (through homotopy methods) (Herings and Peeters, 2010; Govindan and Wilson, 2009). Specifically, equilibrium correspondences, which are set functions mapping games to their equilibria and the integer topological index (lex-index) of connected components of Nash equilibria give both the geometric categorization and orientation and stability, respectively of game equilibria (Balthasar, 2009). These are tantamount to showing how one may then refine or grade equilibria.

Visualizing equilibria via geometric profiles of equilibria in the above fashions may further illuminate game evolution with respect to equilibria dynamics.

Keller and Mulch (2003) introduced multi-agent influence diagrams (MAIDs) in an attempt to better envision and speed up computation of equilibria from the probabilistic dynamics involved in discrete games over the matrix or graph representations of normal and extensive games respectively. Equilibria computational advantageous of MAIDs over game graphs and game trees are highlighted for asymmetric structures and deterministic payoff functions. MAIDs are akin to influence diagrams for Bayesian networks, but generalize them for games. Visualization of probabilistic or uncertainty dynamics (the changes in how probability or other uncertainty distributions or descriptions on pure strategy spaces evolve from information flow) alongside a diagram of how actions and payoffs change in games retains more overall decision analytics for the DM. Topological order is put on the decision of agents based on a relevance graph that is laid on top of the main MAID.

In MAIDs, decision points are depicted by rectangles, chance or cumulative probability distributions (CPDs) variables are denoted by ovals, and utilities are denoted by diamonds. Parent nodes of decisions represent observations, parents of chance nodes represent probabilistic dependencies, and parents of utilities represent parameterizations of those utility functions. Game strategies are then mappings from decision parents (observations) to actions. Pure strategies are as in game theory, a single type of action is picked. Mixed strategies are then probabilistic distributions over those singlet actions. Strategy profiles are the full set of strategies for all agents in the MAID game. Relevancy measures called *s*-reachability are then developed to prune computations for Nash equilibria. This reachability criteria is a probabilistic measure of how decisions made by an agent are relevant or dependent on other decision points.

By connecting together different MAIDs, Gal & Pfeiffer (2008) develop networks of subMAIDs as networks of influence. Each MAID is a node in a network influence diagram (NID) that represents a frame of belief (mental models) of agents. Unfortunately, MAIDs and hence NIDs only target discrete symmetric games with deterministic payoffs even though mention is made of a scheme to include asymmetric games. NIDs also model a way to express quasi-irrationality by letting agents have a positive probability of using different block mental models or MAID nodes in order to make decisions. Coupled with Bayesian network formalisms, NIDs can endeavor to model what we shall call probabilistic irrationality. In this study we consider more general probabilistic models, including quantum, quantum gravity, and classes of paraconsistent logics to direct Bayesian connectives for irrationality potentials. Our interest in MAIDs lies in the graphical economy of its representation of certain game regimes, leading to computational shortcuts based on divide-and-conquer methods for calculating equilibria from smaller subgames which can be expressed as sub-MAIDs. Visualizing general recursive games using equivalent (or nearly equivalent) MAIDs of those games can lead to virtualization of highly dynamic and complex inception games.

While games involve components of risk, visualizing risk has been slow to develop intuitively. Heat maps show color coded clusters of numerically similar groups of information. While risk groups can be presented as such, color is a limited spectra for humans. Risk is in the need for both quantitative atoms of measurement so that those atoms can then be used in visualizations and of ways of displaying such atoms in a meaningful and intuitive manner to maximize the understanding of risk areas. One such approach was from Howard (1984) referred to as *micromorts*. Micromorts (micro mortality) are units of risk that equate to a one-in-a-million chance of a fatality or death occur. The generalization of this are microprobablities which are units of one-in-a-million chances of a general event occuring. This represents an event with probability 10^{-6} of occurring

Gaps in existing research

Simulated inception games interpreted as parallel, fault-tolerant networks and what-if analytical automata has not been proposed, studied or investigated in the existing literature in decision theory, game theory, or causal analysis. Although the inception-like levels in the movie Inception are unlikely to be practical in real world analysis of dreaming, it introduces some intriguing possibilities for executing a coalition game involving computational time warps and gravity, mixed coalitions, multiple nested levels of coercion, and generalized uncertainty and risk. Inceptions may also be a vehicle for building adversarial reasoning methods when dealing with generalized deception, coercion, and diversion in co-opetitive environments with partial or corrupted information. Inceptions, nonetheless, may involve large numbers of agents and a broad diversity of interaction and strategies, utility schema, and uncertainty possibilities. Large equilibria sets may be involved in an inception and hence, the fine tuning of selecting equilibria to use becomes necessary.

In Kott and McEneaney (2007) various authors develop interdisciplinary methods utilizing game theory, AI planning, cognitive modeling, and machine learning for the computation of equilibria or sub-optimal stratagem types of large scale imperfect-information games that are in the form of adversarial reasoning arenas predominantly in warfare, terrorism, and battlefield scenarios post 9/11. Particularly, methodologies utilizing classical coalition game theory, multi-agent machine learning, and stochastic control theory are investigated. Linguistic geometric (LG) methods have been used recently to model large scale and state spaces of multiagent systems as abstract board games (ABGs) (such as chess and tactical theatre warfare), general finite state machines (FSMs) whose structure includes mutual agent influences, using a geometry of states of knowledge representation and reasoning objects known as (LG) zones (Stilman, 2000; Stilman, Yakhnis, and Umanskiy, 2010; Stilman, 2011). LG hypergames are constructed as linked ABGs, followed by the generation of advantageous start states, real-time regeneration of strategies and tactics, and the representation of reasoning of the imperfect information which includes deception (Stillman, Yakhnis, and Umanskiy, 2010). LG hypergames are touched upon in more detail in Appendix B. Inceptions may again be viewed as further abstractions of LG hypergames with more general linkage structures based on the diversity of geometries presentable between inception levels, (i.e., ABGs which are finite state machines, can be generalized to live on non-Euclidean geometries of quantum-gravity structures such as LQG spinfoams and superstrings). In large scale games, Ganzfried, Sandholm, and Waugh (2011) developed techniques for the computation of *strategy purification* and the use of thresholds to form equilibria of strategically similar smaller abstract games that can then approximate the equilibria of a larger game. Mixed strategy equilibria are usurped by more computationally practical and effective nearly-pure or pure strategies for agents. Purification and thresholding for strategies is reviewed in Appendix B. Other adversarial activity in games such as massively multiplayer online game (MMOG) environments has been modeled and distinguished, such as gold farming, in Ahmed, et al. (2011). These behavioral patterns may be generalized in more complex situations involving inception-like scenarios that can be launched in social interactions.

In this study non-classical emergent methods will be discussed for developing inceptions as generalized multi-stage and multi-agency adversarial reasoning machines. Furthermore, it will be posited in this paper that such inception-like level games can lead to the development of certain equilibria strategies in mixed simulation and virtual world emulation of games played by both humans and IQ diverse machines. These results will point to the possibility of constructing such categories of decision games as ways to operate near optimal strategies in complex agenda-based transactions involving the involuntary extraction of information from parties and deceit—the way of modern technical evolutionary survival.

Methodology

The research methodology utilized in this study will be that of a grounded theoretical framing (theory and concept building) based on (i) emergent game theories, (ii) generalized uncertainty metamodels, (iii) the analogies of inceptions as recursive decision branches, (iv) higher level mathematical abstractions of games, and (v) the virtualization and advanced holographic visualizations of risk in decisions and games.

Design as science methodologies will be considered for the simulation-based emergence of models of belief revision within strategic (inception) games ala Markov Chain Monte Carlo (MCMC) simulations of model building of stragegic profile distributions and beliefs about them. In addition, a information science construct of a new model for decision under uncertainty in coalitions of coercion in game situations will be posited. In a followup study to the concepts in this paper , simulation models are constructed, an important artifact of the design of science methodology. The game theoretic model of this study was not constructed based on traditional statistical data analysis as used in quantitative studies or in certain data-centric qualitative studies and their mixed hybrids.

The design of science is a methodological philosophy introduced by Herbert Simon to describe model theory constructs in scientific development (Simon, 1996). For developments in information science several design of science methodology guidelines were developed in order to label examples of such studies (Hevner, March, Park, & Ram, 2004). One of the more important items in these guidelines is that of the production of an artifact model construct. In this study, the theory model constructed is that of a game strategy and structure for the involvement of coercion in information extraction in the scenario of inception-like levels of deception, emulated as dream levels in the movie Inception.

Measurement of the effectiveness of the design of the study model will be a simulation within a virtual world environment played by humans and automaton agents, (i.e., simulation of coercion agents in dream states of deceit under the structure of mixed strategy co-opetive coalition game theory). We will also consider the use of an information criteria (IC) approach to finding optimal (parsimonious and high generalizability) model families and model sizes for assessment pairs for belief revision operations. For families of Bayesian belief models $M\left(\theta_d\right)$, we calculate the following:

$$\begin{aligned} M^*\left(\theta_{d^*}\right) &= \underset{M,\theta_d}{\operatorname{argmax}}\; IC_\lambda\left[M,\theta,N\right] \\ IC_\lambda\left[M,\theta_d,N\right] &= l_N\left[M\left(\theta_d\right)\right] - d\lambda\left(N\right) \end{aligned} \tag{2.1}$$

where $M^*\left(\theta_{d^*}\right)$ is the optimally chosen model, $IC_\lambda\left[M,\theta_d,N\right]$ is the IC statistic to calculate, l_N is the likelihood operator based on the model $M\left(\theta_d\right)$, θ_d is the parameterization of the model of dimension *d*, and $\lambda\left(N\right)$ is a function of the sample size *N*, which characterizes the type of IC statistic to use (Lu, 2011). The AIC (Aikaike) and BIC (Bayesian) information criteria are the most popular versions of these model order selection statistics based on $\lambda\left(N\right) = 2, \log N$ respectively . We choose a consistency

$\left[M\left(\theta_d\right) \underset{P=1}{\rightarrow} M_T, M_T \text{ true model} \right]$ and generalized version of this IC (GIC) from Rao and Wu (1989) where $\frac{\lambda}{N} \underset{N\to\infty}{\rightarrow} 0$ and $\frac{\lambda}{\log\log N} \underset{N\to\infty}{\rightarrow} \infty$.

Simulation Analysis

Simulations of inception games can be gleamed upon to detect patterns of possible stratagem candidates for Nash ε-equilibria and ε-evolutionary behavioral stability. Nash ε-equilibrium will mean that agents will have reached a decision stage point where no further jockeying of change of strategies by all agents, will produce an advantage for any agent within a margin of ε difference in payoffs while ε-evolutionary stability means that no outside agents can eventually be introduced into the game with superior stratagem within ε payoff differences. Note that Nash ε-equilibrium and ε-evolutionary stability generalize the concepts of Nash equilibrium and evolutionary stability respectively, (i. e., ε=0 limiting cases). Inception games inject simultaneous notions of deception, coercion, and persuasion implying the production of intermediate regions of game stratagem instability before eventually converging to ε-evolutionarily stable states and Nash ε-equilibrium. Our equivalent notions of ε -type equilibria is an $\varepsilon -$ inception, the state of an inception team obtaining a nearly guru-consciousness within ε distance.

Patterns of evolution of such game development involving deceit and mutual information extraction under the scenario of inception levels of dreaming will lead to better evolution models for organization negotiation, diplomacy, and coalition building.

Deceit does not have to lead to a limited number of insidiously minded agents having superior positions in a game. This inception model could build on the prospect of mutual growth, despite the deception theme.

Limitations

Since this study is anticipated to be based on proposed game-theoretic structures and models involving inception-like levels of deceit, real world comparison is approximate at best. Similar real world conditions can be emulated in a virtual world game with the inception deceit algorithms to compare some past history of certain events involving espionage, for example. However, these, situations are approximate comparisons and no predictive power is guaranteed or posited. These deceit models are to be used to study past evolutionary developments of some geo-political situations that had involved deceit, in some form. Computational complexity of search and calculation of inception game solutions and equilibria may be NP-hard as in the computation of Bayesian AGM-consistency. The agent size in massive history-based games of which inception games will be framed from, grows the complexity (both computational and storage-wise) exponentially unless certain simplifying structural assumptions are made.

The thresholds for agent like-minded neighborhood size in consideration of Shelling models for inceptions and the distribution of belief revision priors for stragey revision may need to have assumptions made on them for practical modeling. Simulations of human play are gross approximations to the spectrum of human bounded rationality. Strategy regress and the possibility of multiple n-agencies in game play may introduce instabilities into the regions of belief revision. In this study, no human game playing analytics are collected and hence followup studies are needed to reconcile the theory of inceptions with real world human conflict play. Mixing belief uncertainties among agents may lead to a type of *belief neurosis* or further regress in belief revision.

Mapping generalized risk to multiple human senses is limited by current technologies, ethical considerations of pain, and low thresholds of cognitive (sensory) overload. Additionally, few results on the effects of prolonged exposure to holodeck environments and multi-sensorial feedback have been studied. Lastly, this study is a theory-laden endeavor with no prior literature results on inceptions or generalized conflicts modeled on such a conflict structure. No real stable solutions may exists in simulations and there may indeed be a need to make-semi-unrealistic assumptions on the inception model for practical comparisons to real life conflicts to be made.

Implications

It is hoped that the results of this study will have implications for novel approaches in organizational and individual decision behavior when deceit is present in multiple levels of awareness among and across many interacting groups. The application of such interpretation of deceit models of this study may lead to a better understanding of real world diplomacy under the extreme stress of uncertainty and perceived mutual deceit. Generalizations to game theory based on the inception game model may include more consistent time translations between dream levels, asymptotic analysis of inception games in level L_l as $l \to \infty$ and more general rules of engagement for covert information extraction (deceit) so that more general and real world situations can be simulated.

Mathematics of Generalized Inceptions

In a generalization to the inception schema, this study proposes to setup a scenario in which inception-like embedding can be constructed in a virtual world (VW) framework such that persons from different levels of existence can co-exist in one level. One can always know which respective level of existence they are in by having their persona avatars embedded with the level number that they are currently residing in. These numbers are generalizations to the movie's notion of a spinning top in which if the top spins indefinitely, one deduces that they are in a lower level existence, but do not know which level that is. They need to continually spin the top in each ascending level existence. Although one can know of the level they are currently in, they cannot know if other persona are in lower levels in the current level existence. In an alternative version, one can have a probability distribution to assign whether one can know the level number of another persona or of theirs in a particular level existence

Let $vw(i, j)$ depict the level *j* VW of persona *i* for $i = 1, 2, ..., n, \ \ j = 1, 2, ..., m_i$. Hyper-reality for persona *i* is $j = 1$, $\forall i$, (i.e., level 1 existence). Now, $vw(i, j) = vw(k, l),$ for some, $i, j, k, l.$ We assign the inception mapping scheme for persona avatar *i*, as a bi-function, $(f_i, g_i) : (k, l) \to (f_i(k), g_i(l))$. $I_i = (f_i, g_i)$ will depict this inception schema for avatar *i*.

In the probabilistic case of knowing "level consciousnesses" in one's level *j* VW, assign a probability density to each persona avatar, *i*, $p_{i,j}(k)$, which depicts the probability of persona avatar *i* knowing the level of existence of persona avatar *k* in persona avatar *i*'s *j* level VW.

What are the convergence or asymptotic behaviors and strategies of a persona avatar in an inception schema, simulating such evolution in VW environments? Some game-theoretic notions may be used to form a strategy portfolio for a persona avatar *i*, with a particular inception schema I_i.

Definition. Persona avatar *i* is said to reach *consciousness death* if

$p_{i,j}(k) \to 0, \text{ as } j \to \infty, \ \forall k.$

A consciousness death persona is essentially lost in reality with high probability in sufficiently deep levels of existence VWs. They are in an optimal position to be totally gullible.

Definition. Persona avatar *i* is said to reach *consciousness guru* if

$p_{i,j}(k) \to 1, \text{ as } j \to \infty, \ \forall k.$

A consciousness guru eventually reaches the situation where they are nearly fully aware of everyone's existences and their own, superior to other non-consciousness gurus. They are in an optimal position to be omni-influential. These gurus will then possess dominant strategies in a game by the shear weight of their influence.

Consider the normal form of a non-cooperative game structure for coalitions with inception-like properties with time translations for evolutionary qualities. Formally, let *P* be the set of agents, with $n = |P|$, A_i, the set of actions available to agent *i*, and $u_i : A \to \mathbb{R}$, the agent's payoff function, where $A = \bigotimes_{i \in P} A_i$ is the product action set. To form a coalition subgroup of agents of size *m*, one endeavors to obtain a sequence of actions (action vector), $a = (a_i)_{i=1,...,m}$ for that subgroup which are mutually satisfactory. If *P* is partitioned into *N* coalitions, the coalition partition, denoted by $\pi_{P,N}$, will simply be called a coalition structure. Denote by $u_S(a)$ the collective payoff of a coalition $S \in \pi_{P,N}$ with action vector *a*.

Definition. An action vector *a*, is a *coalitional equilibrium* (CE) for *S* and relative to $\pi_{P,N}$ if $u_{S'}(a') \leq u_S(a)$ for any other coalition, $S' \in \pi_{P,N}$ and action vector, a'.

CE are extensions of Nash equilibria in coalition games. In this study, CE will be used as equilibria of interest. CE exist under the following conditions: for each *i*, if A_i is nonempty, compact, and convex, and u_i is continuous and quasi-concave, then for every coalition structure, $\pi_{P,N}$, a coalitional equilibria exists (Ray and Vohra, 1997). Because CE are Nash equilibrium for singleton coalitions, they may also be non-unique (Nash equilibrium may be non-unique). Additionally, transfer of payoffs within a coalition renders more non-uniqueness in equilibrium. A less stringent type of unique equilibrium is that of essentially unique:

Definition. Let $U(\pi_{P,N})$ denote the set of coalitional equilibrium payoffs associated with $\pi_{P,N}$. A coalitional equilibrium is *essentially unique* if $\exists$ sets $U(S,\pi_{P,N}) \in \mathbb{R}^S \ni$ for every $S \in \pi_{P,N}$:

$$U(\pi_{P,N}) = \underset{S\in\pi_{P,N}}{\times} U(S,\pi_{P,N}) \tag{3.1}$$

If a coalition contains an agent that has achieved consciousness guru status separate from others, in an inception rules game, then that coalition may break up based on their respective selfishness. However, a coercion team, entering an inception game as a de facto coalition, may also break up based on higher, more probable payoff structures with other coalition conflicting strategies. The status of consciousness guru either for a group or an individual, usually requires multiple incursions in and out of different dream levels.

In a coalition, the amount of shared resources is what is maximized in a rational strategy. This resource takes the form:

$$R_S = sp_S w_S - \sum_{i\in S} c(r_i) \tag{3.2}$$

where $s = |S|$, $p_s = \dfrac{r_S}{r_S + r_{-S}}$ is the relative frequency probability based on resource shares, r_S is the amount of resources for S, r_{-S} is the amount of resources from all other coalitions, w_S is the value endowment of the collective coalition *S*, and $c(r_i)$ is the individual cost of contributions to *S* (Debraj, 2007). Resources in an inception game can come in the form of consciousness capital, that is, what coalition collectively is more aware of where they are in the dreamscape. This capital for coalition *S*, can be described by the collective sum of probabilities, $r_S = \sum_k \rho_S(k) = \sum_k \sum_{i \in S, j \in S'} p_{i,j}(k)$ where $S' \in \pi_{P,N}, S' \neq S$. One of many possibilities for reshaping coalitions considers multi-coalition formation, (i.e., coalitions that were formally conflicting, merge into one in order to optimize their collective resources *R*, when separately, no clear leader can be sustained). Individually, agents, if conscious of other coalition resources, can jump ship and join another one that would optimize their relative status towards consciousness guru. Hence, for an individual agent, *i*, who is currently in coalition *S*, if $r_{S' \cup i} > r_S$, for some other coalition $S' \neq S$ that agent is considerate of changing coalition membership. However, it is not clear as to what, if anything that agent would lose in such a transition. There could be a backlash of incidents that would affect agent *i*, including the possibility of the new coalition reaching a mega-contract agreement to penalize cross-transfers. That contract may enforce a consciousness penalty (individual transfer of that agent to an unknown dream state, not of their knowing and isolated from all coalitions involved in the mega-contract) to the would-be transfer.

Two major groups are immediately at conflict in an inception, the inception team (coercion team) and the targeted group or individual that extracted information is being coveted from (inceptee team). These two groups represent the first coalitions of a coalition partition in inception games. Physically, only those coalitions can enter the game based on the fail safe nature of the PASIV synchronization device. However, other groups may enter into an inception where members of that inception team and their respective target inceptees share certain information with the first inception game agents. Nonetheless, these two inception games would be mutually exclusive. Information transfer between the two groups of game coalitions is only possible before either engages in their respective inception games.

Inception team coalitions may be broken up by selfish agents or *n*-agencies. However, the targeted (inceptee) agent may also join the inception team to form a high powered coalition provided that the paying client to the inception team is isolated. This then becomes an isolated conflict between the payer client and the targeted agent, turned new payer client. In a more restrictive version of inception, all agreements would be binding and irreversible. Under that scenario, the incursion into multiple dream levels to extract information from the target group will retain coalition structures, with the resource functionals dynamically changing as agents transcend dream levels.

In the inception game and movie, power orders within each of the relevant coalitions (inception team and target team) were important in of the plot unfolded and how the main characters were able to survive their own psychological and physical limitations. Power relations can be investigated within coalition games (Piccione & Razin, 2009). Utilizing strategies that are marked as strongly stable social orders results in long term stability of the order within each coalition, (i.e., leadership is stabilized). In this case, a social power function, q_i assigned to agent *i*, can be emulated using the cumulative consciousness probability function, $q_i = \sum_{l \in S \backslash \{i\}} \sum_{j=1,2.,,,m_i} p_{lj}(k)$. Coalition *S* is more powerful socially than $S' \Leftrightarrow \sum_{i \in S} q_i > \sum_{i \in S'} q_i$. Denote by $\Sigma_\rho = \{S_{i_1}, ..., S_{i_N}\}$ to be the social order of the coalitions in the partition, $\pi_{P,N}$ according to the vector of social order functions, $q = (q_1, ..., q_n)$. Let $V_i(\Sigma)$ be the social rank of the agent *i* induced by the social ordering, Σ.

Def. A deviation or shift from the social order, Σ, by a subset of agents, *C*, will result in a reordering of the social order as $\Sigma_C = (C_1 \setminus C, ..., C_n \setminus C)$. A deviation social order, Σ_C from Σ, induced by *C* is *profitable* if $V_i(\Sigma_C) < V_i(\Sigma)$. Define a mapping, $\partial : \mathrm{E} \to P(n)$ between the set of all social orders, E, and the power set of agents, *P* such that $C \in \partial(\Sigma) \Leftrightarrow$ (i) *C* is a profitable deviation from Σ and (ii) there does not exist $C' \in \partial(\Sigma_C)$ such that (a) $C \cap C' = \varnothing$ and (b) $V_i((\Sigma_C)_{C'}) > V_i(\Sigma)$, for some $i \in C$. A deviation of social order induced by *C* on Σ is *durable* if $C \in \partial(\Sigma)$. It has been shown by Piccione and Razin (2009) that ∂ is a well defined mapping. Finally, a social order $\Sigma = (C_1, , , C_n)$ is *strongly socially stable* if $\partial(\Sigma) = \varnothing$ for all power social relations, *q*.

Strong social stability is akin to Nash equilibrium because agents are deincentivized to change allegiances in order to climb their relative social order due to no advances in profitability (inception relative consciousness) scaling for their team. One may then consider condition(s) on the consciousness probabilities, $p_{i,j}(k)$ where an inception game can be considered strongly socially stable and hence, in the long run, stable with respect to leadership roles and the goals of achieving or preventing inception in the various coalition teams traveling through the inception dreamscape.

Agents within a coalition, can use individual mixed strategies or coalition mixed strategies that are correlated (are in tune to each other's strategy) in the sense of randomizing the next strategy that they will use. If agents use individual mixed strategies, then it is not guaranteed that a CE exists (Haeringer, 2002). Therefore, coalition correlated strategies must be used in order to have a chance at reaching a CE and therefore, a situation where coalitions will no longer be changing strategies towards their respective inception goals in inception games.

In inception games, the inception team is endeavoring to extract or influence a change of decision-making on behalf of the incept team or individual. In an epistemic logical approach to inception, we may invoke the processes of belief revision logical operators (revision, expansion, and contraction) so as to introduce changes to or novel injection of beliefs in the logic preference of the inceptee in such as way as to minimize the amount of possible added logical contradiction in the knowledge base of beliefs of that inceptee, thereby conservatively preserving as much of the original inceptee's belief world as possible (Alchourron, Gärdenfors, & Makinson, 1985; Gärdenfors, 1992; Ribeiro, 2013). Minimal amounts of deletion or change of epistemic propositional beliefs should transpire due to the potential cost of loss of valid collateral information. Inceptions may become more covert than overt in that case.

Let Δ denote a logic system. Belief revision (change) comes in three distinctive forms: (i) expansion: an added proposition (contents of a sentence) ϕ, is added to Δ without regards to inconsistent consequences and is denoted by $\Delta+\phi$, (ii) revision: a new proposition ϕ is added to Δ but makes it inconsistent , some sentences are then deleted to make it consistent and this new logic system is denoted by $\Delta\dot{+}\phi$, and (iii) contraction: a set of sentences in Δ given by $\phi(\Delta)$ are removed without adding ϕ in order to make Δ consistent with ϕ and this is denoted by $\Delta\dot{-}\phi$ (Gärdenfors, 1992).

Trivially, the construction of $\Delta+\phi$ is practical through simple logical operators. Revision and expansion present with problems not solvable by direct application of logical operators. Algorithms akin to belief change operators φ,must be applied to Δ based on conditions that approximate or judge rational consistency, at least for rational consequences. The emergent systems considered in this study broaden this approach to consider boundedly rational and spectral rationality under general non-classical physical and logical systems. Gabbay, Rodriques, and Russo (2007) consider belief revision operators for first-order non-classical logic systems and hence for paraconsistent logics that may refine or change rationality definitions in classical logics due to their preservation of inconsistent and paradoxical logic systems such as the family of LP logics. Quantum superposition through a strong Zermelo-Franckel (ZF_1) set theory, and other subclasses of Kripke Algebraic logics through other aspects of quantum systems, generate paraconsistent logic systems (Luisa, Chiara, and Giuntioni, 1989; Da Costa and De Ronde, 2013). More recently, a parametrization of quantum logics referred to as paraquantum logics (PQLs) using a logical state ψ dependent on the propagation of the degrees of evidence of the measurement stage of a quantum system, was developed (Da Silva Filho, 2011). Similarly, causaloid probabilities developed in Hardy (2007), discussed in more detail in Appendix D and injected into the generalized framework of emergent versions of inception games in this study, depend on large-scale quantum-relativistic probabilities across vast event spacetime regions and can be considered as paraconsistent non-temporal logics because spacetime is treated as a single manifold for indefinite causal structures without a separable time subdimension. Booth and Richter (2012) present and display the case for fuzzy belief revision utilizing generalized Tarski deductive and Lulasiewicz fuzzy logics. Zadeh's GTU representations subsume these logics by their higher level constraint-based precisiation-language constructs. Hence, here one may consider GTU-based belief revision systems and spaces of belief revision operators φ_g based on GTU constraint objects g, including higher-order logics that would include higher-order paraconsistent fuzzy logics.

We briefly mention the components involved in our generalized inception game. The components of an inception game will include: (i) coalition teams/agents (inceptors and inceptees) with possible reverese or anti-inception scenarios, (ii) inception information silos, (iii) payer source (client) and a payoff structure for agents to be distributed, (iv) inception dream levels and physical/psychical rules of engagement (time-discounted recursions), (v) consciousness awareness thresholds for inception adcantage, (vi) uncertainty structure (generalized stochastics) for consciousness awareness and social power status for agents, (vii) risk profiles (spectral range) of agents and collective coalitions and risk neighborhood size thresholds, and (viii) coalition bonding thresholds (propensities of agents to *n*-agency behavior).

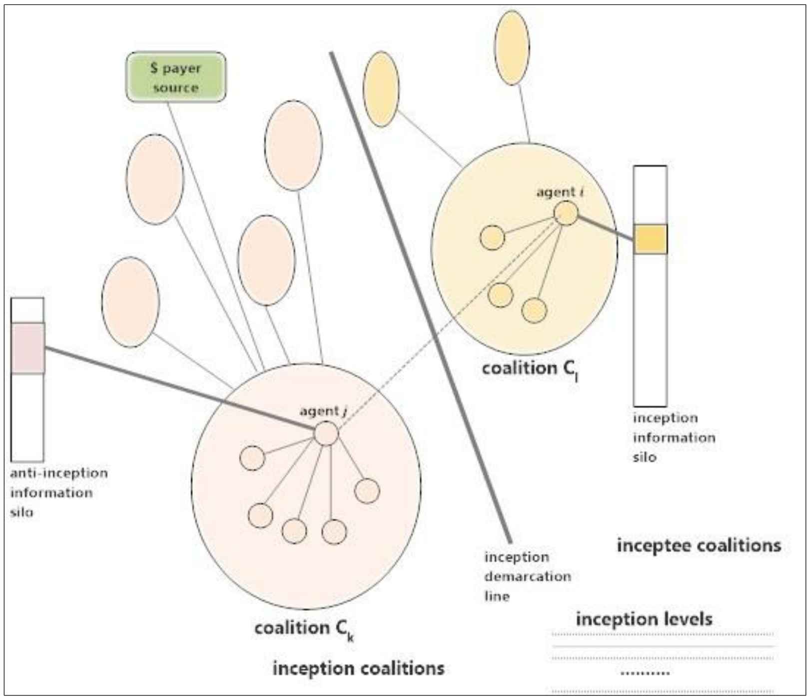

Figure 2 - Components of General Inception Game

While inceptions are manifested by chain-events of possible belief revision operators changing the knowledge base belief system of the inceptee through the course of navigating inception levels, the target event is one of the exposure of a large enough chunk of insider information in those knowledge bases. Within the resources that each agent possesses R_i, including shared coalition-level resources, $\left\{R_{C_j}\right\}_{j=1,\ldots,M}$ where *M* is the number of coalition teams in an inception game, this inception information chunk given by Z, is buried in and can be expressed as a subset of $R=\bigcup_{j=1}^{M} R_{C_j}$. If the inception game is additive in coalitions, (i.e., the resources within a coalition team are equal to the sum of the individual coalition agent resources within that team) then $Z \subset \bigcup_{i=1}^{N} R_i = \bigcup_{j=1}^{M} R_{C_j}$ where M is the total number of agents. Under this additivity condition, let $Z_i = Z \cap R_i, i = 1, ..., N$ be the individual agent resource contributions to the totality of inception information Z. Agent i with a social power (conscious-awareness rating) weight given by a social preference order, can coerce their respective part of inception information Z_i, proportionally with respect to other agents. Denote the agemt and coalition payoffs respectively as,

$$
\begin{aligned}
U_{C_j}(A) &= \sum_{\substack{i \in C_j \\ a_i \in A_i}} {}^{a_i}Z_t^{C_j} g(a_i) \\
U_i(A_i) &= \sum_{a_i \in A_i} {}^{a_i}Z_t^{i} g(a_i)
\end{aligned}
\tag{3.3}
$$

where ${}^{a_i}\mathrm{Z}_t^{C_j} = \bigcup_{k \in C_j} {}^{a_i}Z_t^k,\ {}^{a_i}Z_t^i \subset Z_t \cap f_i(R_t^i),\ Z_t \subset f(R_t), g(a_i)$ is a GTU-based uncertainty operator on action a_i, $f_i : R_i \to Z,\ f : R \to Z$ are extraction functions acting on agent resources producing information subsets, ${}^{a_i}Z_t^i$ is the extracted inception information as a result of action a_i at time t (stage) for agent i, and ${}^{a_i}Z_t^{C_j}$ is the extracted inception information as a result of action a_i at time t (stage) for coalition j.

For the proceeding discussion on belief revision operations in inceptions we adopt the history-based version of gasmes structures where all agents history of action moves are recorded and stored as part of agent information sets. Infomration sets are the total of an agent's information about the game and the circumstance surrounding that gaem including all histories of agent movement up to that point in a game stage, payoffs, and agents. In a traditional general decision branching tree agents make moves sequentially. A particular branch path of an decision tree represents a history of agent moves. Decision branching trees may be translated to their analogous extensive-form game trees where payoffs for every agent strategy combination are displayed in general hypermatrices in the multi-agent case. Information sets are then the sum total of an agent's knowledge of agent's histories, payoffs, and game structure up to that point in time (game stage). These information sets consist of individual information subsets that display the agent payoffs for a particular history path. Belief revision operators, as discussed in this study, will then be applied to these information sets which represent agent belief systems about the game.

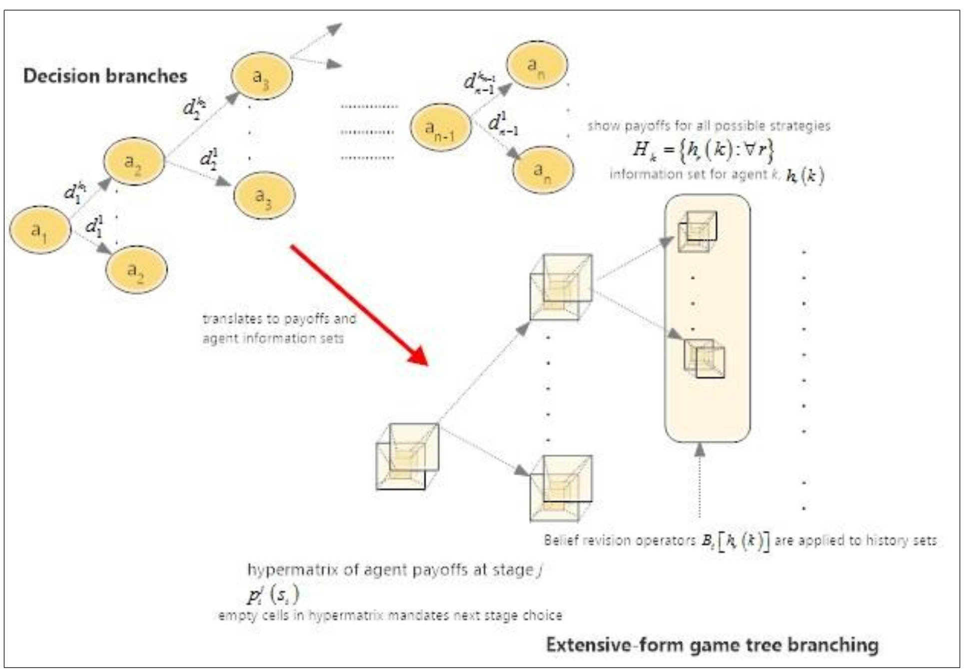

Figure 3 - Decision and Game Tree Branching

Belief revision of agent strategies are applications of belief revision operators on the belief systems of the opposition in inceptions. General belief revision are compositions of single belief revision operators. Each agent action can then be equted with a composition of belief revision operators,

$$a_i \leftrightarrow \left(\varphi_i^j\right) \equiv \varphi_i^{j_1} \circ \ldots \circ \varphi_i^{j_k} : B_{C_j} \to B_{C_j}^u \tag{3.4}$$

Strategies are mixed action profiles through stages of game transitions $s_i\left(b_1,\ldots,b_k,\ldots\right)$,

where b_k is the k-th stage profile action as a GTU-based mixture $b_k = g\left(A_i\right) = \sum_{j=1}^{L_k} w_j g_j\left(a_j\right)$,

for some weighted sum of GTU distributed actions in a simple linear case. These epistemic belief systems in games can be viewed as agent beliefs about past histories of moves. These past decision histories H_i, are a component of finite extensive games, along with an equivalence relation for those histories $\left\{\approx_i\right\}_{i\in N}$ that define history classes where actions are taken without regard to a history in that class.

We closely follow the development of AGM-consistent belief revision from Bonnano (2011) before extending those results to the more general case of GTU-based uncertainty operator revision in inception games. Define an assessment as a pair $\left(\sigma,\mu\right)$, where σ is a strategy profile and μ is a belief system on those profiles. Assessments are the basis for updatable epistemic games. Belief systems are updated using a Bayesian procedure or GTU-based updating scheme that generalizes Bayesian updating by the information set I_i, of the agent's accessible nodes in the equivalent game decision tree.

Each information set I_i, denotes the sum total of the agent's knowledge of the universe. Nonetheless, compatibility between the strategy profiles and the belief system about those profiles must be present for a consistent approach to epistemic game solutions and revisioning. Compatibility between σ and μ can be quantified by the concept of *KW-consistency* - there exists an infinite sequence of mixed strategy profiles $(\sigma_i)_{i=1,\ldots}$ such that $\lim_{i\to\infty}(\sigma_i, \mu_i^u) = (\sigma, \mu)$ where μ_i^u is the updated belief system of μ_i which is associated with the strategy profile σ_i (Kreps and Wilson, 1982). More recently, a more practical type of assessment compatibility was developed, the AGM-consistently, which is associated with the idea of the semantically powerful and minimally invasive AGM belief update revision framework (Alchourron, Gärdenfors, and Makinson, 1985). AGM-consistency is based on a *plausibility order* on the space of all agent histories of actions $H = (H_i)_{i=1,\ldots,N}$ as a means of imposing a total pre-order on comparing those histories, (i.e.,binary relations $\precsim \subseteq H \times H$ which are complete and transitive).

Def. The assessment (σ, μ) is *AGM-consistent* if (i) $\sigma(a) > 0 \Leftrightarrow h \sim ha$ [w.r.t $\precsim$] and (ii) $\forall h \in D,\ \mu(h) > 0 \Leftrightarrow h \precsim h', \forall h' \in I(h)$, where $\sigma(a)$ is the probability of action a as assigned by σ, *ha* is the history sequence *h* followed by action a, $\mu(h)$ ia the probability of history *h* as assigned by μ, and $I(h)$ is the set of histories with the same information set as *h*.

If $\precsim$ makes the assessment (σ,μ), AGM-consistent, then $\precsim$ is said to *rationalize* (σ,μ).

(σ,μ) is said to be *Bayesian relative to* $\precsim$ if for every $\precsim-$ equivalent class *E*, there exists a probability measure $\upsilon_E : H \to [0,1]$ such that (i) $Supp(\upsilon_E) = E$ and (ii) if $h, h' \in E$ and *h* is a prefix of h' then $\upsilon_E(h') = \upsilon_E(h) \times \sigma(a_i) \times \ldots \times \sigma(a_m)$ and (iii) for every information set *I* such that $Min_{\precsim} I \subseteq E$ and for every $h \in I, \mu(h) = \upsilon_E(h \mid I) = \frac{\upsilon_E(h)}{\upsilon_E(I)}$ where

$$Min_{\precsim} I = \left\{h \in I : h \precsim h', \forall h' \in I\right\}.$$

Def. An assessment (σ,μ) is *Bayesian AGM-consistent* if it is rationalized by a plausibility order $\precsim$ and it is Bayesian relative to $\precsim$.

Def. (σ,μ) is a *perfect Bayesian equilibrium* if it is Bayesian AGM-consistent and sequentially rational.

Perfect Bayesian equilibrium $\Rightarrow$ subgame-perfect equilibrium and sequential equilibrium $\Rightarrow$ perfect Bayesian equilibrium. Hence perfect Bayesian equilibrium is intermediate between subgame-perfect and sequential equilibrium.

Result: Every finite extensive-form game *G* (and hence every finite decision game tree $\Gamma(G)$ based on *G*) has at least one perfect Bayesian equilibrium assessment (σ, μ).

Belief revision operators φ, act on propositions (sentence contents) in a belief (logic) system B, by one of three broad categories: (i) expansion denoted as $B+\varphi$, (ii) revision, $B\dot{+}\varphi$, and (iii) contraction, $B\dot{-}\varphi$. Revision and contraction require belief revision operators to be minimally invasive (i.e., conserve as much as possible, logical consistency within the updated belief system to B, denoted as B^u, according to the epistemic AGM-consistency framework. Let Φ denote the set of formulas in a propositional language *L* which is based on a countable set of atoms, *S*. Subsets $K \subseteq \Phi$ have deductive closures denoted by $[K]$. K is closed if $[K] = K$ and is consistent if $[K] \neq \Phi$. Formally, an agemt's initial belief system is consistent and closed but is exposed to subsequent information given by $\Psi \subseteq \Phi$. A *belief revision function based on K* is a function $B_K : \Psi \to 2^{\Phi}$ such that $B_K(\phi) \subseteq \Phi, \phi \in \Psi$. If $\Psi \neq \Phi$, then it is a partial belief revision. Otherwise, if $\Psi = \Phi$, it is a full belief revision. If B_K satisfies the aGM postulates as given in Alchourron, Gärdenfors, and Makinson (1985), then it is an *AGM belief revision function*. Nonetheless one must reconscile belief revision syntactically in propositional languages with that in set-theoretic game structures. Choice frames (from rational choice theory) bridge these two concepts and serve as the link needed for our treatment of generalized belief revision systems.

Def. The triplet $C=(\Omega,\mathcal{E},f)$, where $\Omega\neq\varnothing$ is a set of states and subsets of them are events, $\mathcal{E}\subseteq 2^{\Omega}$ is a collection of non-null events in Ω, and $f;\mathcal{E}\rightarrow 2^{\Omega}$ is a function the associates events with non-null events $f(E)$ satisfying $f(E)\subseteq E$, is called a *choice frame triplet*. Interpret $E\in\mathcal{E}$ as the available alternatives (potential information) and $f(E)$ as the chosen alternatives which are doxastically (believably) possible.

We define a condition to be imposed on the histories of agents.

Condition C: $\forall i\in N,\forall h\in D,\forall a\in A(h)$, if $h\in D_i$ then $ha\notin D_i$, where D is the profile decision space for all agents and D_i is the profile decision space for agent *i*, (i.e., no consecutive actions by any agent). In the case of inceptions, one can compose a series of actions into one actionable move.

We now associate models with choice frames through valuations of atomic formuli $v:S\rightarrow 2^{\Omega}$, which map formuli to the states that are true under them. A model is then the quadruple $M=(\Omega,\mathcal{E},f,v)$ and is an interpretation of the choice frame $C=(\Omega,\mathcal{E},f)$.

Def. A choice frame $C=(\Omega,\mathcal{E},f)$ is *AGM-consistent* if for every model *M* based on it, partial belief functions B_{K_M}, associated with *M*, can be extended to full belief functions that are AGM belief functions. Here $K_M = \left\{\phi \in \Phi : f(\Omega) \subseteq \|\phi\|_M\right\}$,

$\|\phi\|_M =$ {truth set of formulas $\{\phi \in M\} = \{\omega \in \Omega : \omega \models_M \phi\}$} and $\omega \models_M \phi$ means that ϕ is true at state ω in model *M*. $C = (\Omega, \mathcal{E}, f)$ is *rationalizable* if there exists a total pre-order $\precsim$ on Ω such that for every $E \in \mathcal{E}$, $f(E) = \left\{\omega \in E : \omega \precsim \omega', \forall \omega' \in E\right\}$, where the total pre-order $\omega \precsim \omega'$ is interpreted as ω is at least as plausible as ω'. $f(E)$ is thus the set of most plausible states in *E*.

One may then build partial belief revision functions as:

$$B_{K_M} : \Psi_M \to 2^{\Phi}, \Psi_M = \left\{\phi \in \Phi : \|\phi\|_M \in \mathcal{E}\right\}, B_{K_M}(\phi) = \left\{\psi \in \Phi : f\left(\|\phi\|_M\right) \subseteq \|\psi\|_M\right\} \qquad (3.5)$$

Result: $C = (\Omega, \mathcal{E}, f)$ is AGM-consistent $\Leftrightarrow$ C is rationalizable.

Now let $\mathcal{P}_i$ denote the set of total pre-prders that rationalize a game choice frame $\left\{(H, \mathcal{E}_i, f_i)\right\}_{i \in N}$ and additionally satisfies condition PL1 and PL2i of Bonnano (2011). Define a game common prior by $\mathcal{P} = \bigcap_{i \in N} \mathcal{P}_i$ (common initial beliefs and disposition to change those beliefs in a game setting). Here *H* denotes the agent game histories of strategy movement and $\left\{(\mathcal{E}_i, f_i)\right\}_{i \in N}$ denotes the belief and dispositions to change those beliefs about those game strategies.

Result: A game choice profile given by $\{(H, \mathcal{E}_i, f_i)\}_{i \in N}$ admists a game common prior if there exists a total pre-order $\precsim$ on H that rationalizes the belief of all agents and satisfies the conditions PL1 and PL2i for each agent, (i.e., $\mathcal{P} \neq \varnothing$).

Main result: Let an extensive form game (and hence a game tree) satisfying condition C be given by $G\ (\Gamma(G))$. Let $\{(H, \mathcal{E}_i, f_i)\}_{i \in N}$ be a profile of AGM-consistent choice frames for the initial beliefs and disposition to change those beliefs for all agents in *G*. If $\{(H, \mathcal{E}_i, f_i)\}_{i \in N}$ admits a common prior, (i.e., $\mathcal{P} \neq \varnothing$) then every common prior $p \in \mathcal{P}$, is a plausibility order and hence is a belief revision operator for *G*.

More generally, belief revision operators based on GTU constraints *g*, denoted by φ_g, acting on inception game belief systems *B*, can represent very general belief revision uncertainty schema including Dempster-Shafer, Zadeh possibility distributions, quantum probabilities, quantum-gravity causaloids, Bayesian causal nets, fuzzy belief functions, rough set approximated belief functions, classical probabilities, first-order and higher-order logics, including paraconsistent systems. Composed GTU-based belief operators can then be constructed to form generalized belief revision operators where each component operator satisfies conditions of AGM-consistency and existence of common priors as above, resulting in generalized game agent actions of the form:

$a_i(\bar{g}) \leftrightarrow \varphi_i^{j_1}(g_1) \circ ... \circ \varphi_i^{j_k}(g_k) : B_{C_j} \to B_{C_j}^u$, where $\bar{g} = (g_1, ..., g_k)$ is a GTU-based constraint vector scheme for *k* cascaded (composed) agent behavior revision operators.

We now investigate a separate general statistical mechanism for updating beliefs about agent strategy profiles where GTU-based operators are utilized instead of strictly classical probabilities. Agent belief systems *B*, which are knowledge bases of propositions, may be updated more precisely in a statistically robust manner, based on GTU-based belief revision operators φ_g, and a finite sample of *N* experimental runs, using generalized likelihood-type transformations π_{φ_g}, that in turn, utilize updated information at stage *k*, given by I_k, as in Bayesian approaches to forming posteriori probability distribution. Propositions $\phi \in B$, which may be treated as general uncertainty distributions or rules in and of themselves, are transformed to another rule/distribution proposition $\phi_{\varphi_i(g,\theta)}^N$, by the likelihood operator that is based on a GTU-based distribution/rule $\varphi_i(g,\theta)$, for agent *i*, and *N* samples, $\pi_{\varphi_i(g,\theta)}^N \phi \to \phi_{\varphi_i(g,\theta)}^N$. In this formalism, $\varphi_i(g,\theta)$ entails a general object parameterization θ.

Chained compositions of belief revision operators can be interpreted as recursive likelihoods, (i.e., given updated information in each inception level, a chain of uncertainties about uncertainties are generated). We refer to these chains as *higher-order beliefs*. Inception strategies are then profiles of GTU mixed actions whichas chained compositions of GTU-based likelihood-type transformations of belief system propositions.

Given a threshold η, defining a safety margin from consciousness awareness advantage of one agent over another or of a coalition agency, computing inception game equilibria (or any other equilibira type) resembles a Schilling type of model of segregation model dynamic (Schilling, 1969). More recnt results show that segregation or end game social habitation convergence depends on neighborhood sizes and psychological segregation thresholds of happiness (Barmpalias, Elwes, and Lewis-Pye, 2013). See the section on morphogenesis approaches to game equilbira in this study for details. Inception neighborhoods are those areas that influence an agent directly from individual agents and coalition inceptions and information exchanges from such phenomena.

Agent belief operators are dependent on agent (and coalition) social power which is indicated by consciousness-awareness, as defined earlier in this study. To review, consciousness-awareness is the relative ability to know that you are in a certain conscious (inception) level, while others do not. We now generalize our earlier definition of probabilistic consciousness-awareness to GTU constraints G_i for an agent *i*:

$$\begin{aligned} G_i(j,k) = &\text{ GTU uncertainty that agent } i \text{ knows consciousness awareness} \\ &\text{of agent } k \text{ in agent } i\text{'s inception level } j \end{aligned} \tag{3.6}$$

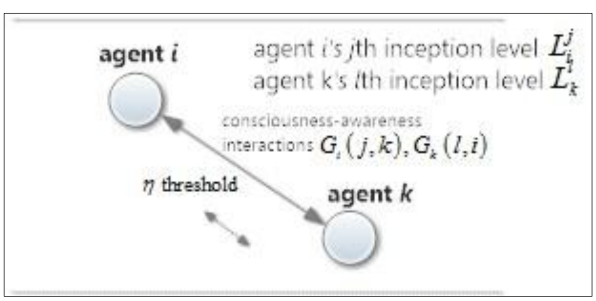

Figure 4 - Agent consciousness-awareness interaction

Agent *i* revises the belief system of agent *k* in inception level *j* if and only if $G_i(j,k) > G_k(j,i) + \eta$ for some threshold η. $\{G_i(j,k)\}, \forall(i,j,k)$ induces a preference ordering for belief revision operations and hence for strategy execution. Bawed on social power preference, a weighing of influence will affect extraction of the portion of the inception information from an agent's influence neighborhood through belief revision operators acting on the inceptee's belief system and hence on the consciousness-awareness uncertainties $G_i(j,k)$. On then tests for the threshold condition:

$$a_i(\bar{g}) \equiv \varphi_i^{j_1}(g_1) \circ \ldots \circ \varphi_i^{j_v}(g_v) : G_i(j,k) \to G_i^{\bar{g}}(j,k) \overset{?}{>} G_k(j,i) + \eta \tag{3.7}$$

The knowledge that an agent is aware of an inception attempt happens when the probability (GTU uncertainty) of consciousness awareness over another agent is within a threshold η (but not greater) and other agents attempt inceptions against that agent:

$$G_k(j,i) < G_i(j,k) < G_k(j,i) + \eta \tag{3.8}$$

This situation is akin to asymmetrical agent information, biases against any attempted belief revision, no matter the truth value, T (or truth membership value) of the belief revision operator(s) update on *B* of the inceptee. Hence, the aprior agent knowledge of an outside inception weighs against any belief revision attempt, regardless of belief revision update AGM-consistency, as discussed earlier. On the other hand, if an inception is anticipated, but no such attempt is actually made, the agent is less likely to accept useful information from other agents in information exchanges pertaining to other inceptions (intra-coalition and favorable *n*-agency exchanges).

Epistemic effects from inceptions include truthfulness vigilance – the propensity to initially trust and follow through with stronger verification in the later stages of interaction and decision-making regardless of trusting history. Gilbert (1991) posited that a Spinozoan unity doctrine $\mathfrak{S}$, of initial belief in a proposition is necessary in order to commence understanding (and veracity) of that proposition in a belief system. Moreoever, in general, initial doubt or unacceptance of a proposition is harder to materialize than its initial acceptance. More recent research has shown that a suspended belief – a conditional Cartesian doctrine $\mathfrak{C}$ – is more likely in an initial understanding of propositions with informative priors (Hasson, Simmons, and Todorov, 2005). Hybrid Cartozan frameworks $\mathfrak{S}\text{–}\mathfrak{C}$, in which comprehension followed by temporary acceptance and then possible rejection are also posited. Epistemic vigilance may be more likely a scenario following an initial acceptance after comprehension. Here, we propose that a recursive evolutionary process takes place that manifests in possibly chaotic or recursively fractal regions of attraction between $\mathfrak{S}$ and $\mathfrak{C}$ using belief evision meta-theories with GTU-based belief revision oeprators φ_g, as discussed before. We choose GTU operators for the atomic operators of belief change because of their generality in representing vast diversities of logic systems for uncertainty.

Understanding a proposition is more mechanical (e.g., computing complexity measures) than believing it (e.g., surmising if a belief proposition is AGM-consistent within a belief system). We consider the inception model as a meta-type for a game involving recursive belief revision through time-discounted inception levels (to be discussed in the next section) and application of GTU-based belief revision operators to the belief systems of coalition agents involved in an inception attempt. Coalitional belief equilibria are possible under conditions of informative common priors on agent initial belief systems and dispositions to change those beliefs through assessments (σ, μ), which are Bayesian AGM-consistent within the extensive form of an inception game. The concept introduced in this study known as $\varepsilon-$inceptions, are the more likely behavior solutions in inceptions where perturbations between regions of $\mathfrak{S}$ and $\mathfrak{C}$ doctrinal behaviorial tendency on beliefs of common priors. $\mathfrak{S}-$ prior beliefs lead to belief revision operators that initially support, with certitude, the truth values of the status quo propositions being challenged in inceptions. $\mathfrak{C}-$ prior beliefs lead to initial support, with certitude, the false value of status quo propositions in inceptions.

In a GTU-based belief revision regime, a spectrum of initial belief in status quo propositions in inceptions is created based on the common prior distributions' informativeness of those propositions. We denote thos spectral measure of epistemic belief doctrines by ζ. This measure may be a complex of objects rather than a simple scaled number. We can then ceate an approximate simplifying normalized number based on this complex $S-C(\zeta)$, that measures a divergence of the ensuing initial prior support for status quo propositions that goven agent beliefs about other agents' strategies, ranging from an $\mathfrak{S}$ to a $\mathfrak{C}-$like doctrine.

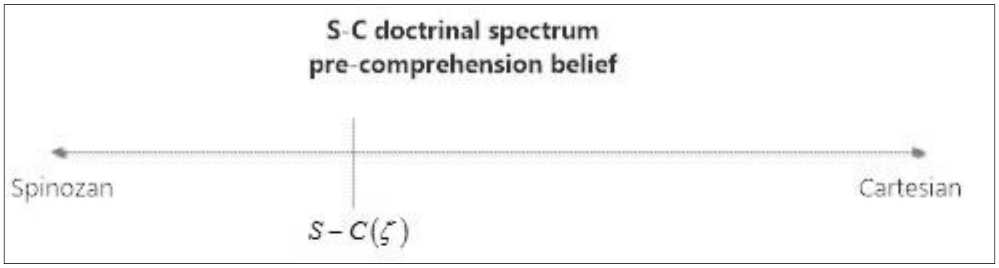

Figure 5 - Spinozan-Cartesian Spectrum

In infinite games such as conceptual inceptions, strategy near-solutions may cycle in regions in the S-C spectrum. The inception information sets of the agents may dictate certain regions of attractions and hence of certain beief doctrin spectral subregions, Common informative priors of agent s are candidates for inceptions and for manifesting well-defined belief doctrine spectral measurements. Nonetheless, how would one measure belief revision potential on the S-C spectrum? In inceptions, the end game value (payoff) is the information that is sought after and an ensuing chaning of the “hearts and minds” of the inceptee. We assume the initial belief systems of the coalition agents are the target. Belief revision operators are applied to those belief systems. The common priors to those initial beliefs are then the baseline for belief revision movement. Belief revision operators or subsequent priors to an agent’s belief system then result in an updated agent belief system. In a Spinozan doctrine, those belief revisions are not changed as much (if at all) as those from a Cartesian doctrine because acceptance of the initial beliefs is more likely froma Spinozan than a Cartesian or evn a Cartozan. Therefore, a metric that would measure a distinction of this initial movement would be a divergence measure between priors after and before a belief revision operation. A candidate for this divergence would be a Csiszar *f*-divergence which is a generalization of the KL-divergence between probability measures. We generalize this divergence to measure differences between GTU-based uncertainty operators. Define a Csiszar-Morimoto-Ali-Silvey (CMAS)-generalized divergence between an initial prior p and its belief revision update p^u for each $p \in \mathcal{P}$:

$$D_g^{f,w}\left(p, p^u\right) = tr_g\left[wf\left(\frac{dp}{dp^u}\right)dp^u\right] \tag{3.9}$$

An accumulated (total) divergence between prior spaces $\mathcal{P}$ and $\mathcal{P}^u$, may then be formed as:

$$^T D_g^{f,w}\left(\mathcal{P}, \mathcal{P}^u\right) = \sum_{p \in \mathcal{P}} tr_g\left[wf\left(\frac{dp}{dp^u}\right)dp^u\right] \tag{3.10}$$

We symmeterize this total divergence using the weighted Jenson-Shannon divergence scheme to obtain a final metric for prior spaces:

$$D_g^{f,w}\left(\mathcal{P}, \mathcal{P}^u\right) = \sigma\, {}^T D_g^{f,w}\left(\mathcal{P}, \mathcal{P}^u\right) + \left(1-\sigma\right) {}^T D_g^{f,w}\left(\mathcal{P}, \mathcal{P}^u\right) \tag{3.11}$$

for $\sigma \in [0,1]$. We now define a unit norm on the space of possible belief priors Π:

$$\left\|\mathcal{P}\right\|_\Pi = \frac{{}_\sigma^T D_g^{f,w}\left(\mathcal{P}, \mathcal{P}^U\right)}{\sup\limits_{\mathcal{Q} \in \Pi \setminus \{\mathcal{P}\}} \left\{ {}_\sigma^T D_g^{f,w}\left(\mathcal{P}, \mathcal{Q}\right)\right\}} \tag{3.12}$$

where $\mathcal{P}^U$ is the belief system updated after uniform priors are applied as belief revision operators to all individual agent priors in $\mathcal{P}$. One may then define a normed unit spectral measure for $\mathcal{P}$ on $S - C\left(\zeta\right)$ by applying the unit norm above to get,

$$\zeta_{\mathcal{P}} = \left\|\mathcal{P}\right\|_\Pi \tag{3.13}$$

so that for any $\mathcal{P} \in \Pi$, $0 \le \zeta_{\mathcal{P}} \le 1$.

In our discussions of belief revision, we concluded with the dependence of epistemic common priors existing as one major condition for belief revision to be plausible. Bayesian updating of GTU distributions remained the mechanism for these updated revisions. However, Bayesian priors remain controversial with respect to the ultimate arbiter of who revises or chooses them in the updating scheme and of their overall effectiveness and potential overconfidence exercised by Bayesian decision-makers (Bowers and Davis, 2012a; 2012b; Griffiths, Chater, Norris, and Pouget, 2012). Given that Bayesian updating using reliable consecutive priors leads to more accurate prognostication based on the asymptotic probabilistic robustness of those updated priors, calibration of those priors and hence, of the arbiter of those priors is necessary to retain a convergence to truer models of reality. In this section we iterate some recent discussion on the calibration process for Bayesian decision-makers. Mannes and Moore (2013) in a recent series of experiments conclude that underscoring the precision of decision-making agents from past histories of updating revisions improves the progrnostication record of those DMs. Additionally, they point to the introduction of certain amounts of alternative beliefs in the space of priors as a way of normalizing confidence. Inception priors may be more precise by flattening agent prediction egos through these two calibration correction techniques. The use of a diversity of uncertainty operators for a larger amount of situational epochs, (i.e., the use of the full power of GTU constraint representations, mixing,a nd application) may also be advised for inception-like games of conflict in order to achieve a modest amount of probability calibration for DMs.

Calibrated decision-makers lead to better measurements of risk. To this end, we need to form a real-time or stage-updated comprehensive and multi-dimensional risk measurement object during game play that takes into account aspects of uncertainty operators and scenarios, more complete information sets, and consistent calibration of agents. This risk object must be communicated to the DM in a more natural, ubiquitous, and instantaneously comprehensive manner. This is the subject of the third part of this study, a more effective means to convey risk via multi-sensorial mappings of risk components to the sensorium of DMs. Risk measurement and visualization in inception strategy dynamics involve an overall risk metric or manifold $\mathfrak{R}$ that is a multi-dimensional assessment of risk related components which include: (i) expcted utility values at stages or time epochs (w.r.t. GTU-based uncertainty operators), (ii) psychologically assigned belief weights toutilities, uncertainties, and payoffs, (iii) coalition partition preferences (who do you want to prosper or lose reagardless of your situation), (iv) risk tolerance – thresholds on a spectrum of risk-aversion/aggression (fear/confidence), (v) statistical and computational error tolerances of quantitative risk measurements, (vi) targeted Lipschitz game effects on agents or groups (similar to iii. above), (vii) size and makeup of coalitions, (viii) risk capacity – th relative resource reservoir (loss absorption capacity), (ix) time horizon of play endurance (time expanded fluctuation and volatility smearing), (x) risk efficiency (vs. utility), also called *efficiency frontier* (modern portfolio theory

[MPT] curve of optimal utility vs. risk), and (xi) confounded effects between the above factors, (i.e., resource size vs. risk tolerance).

We propose a novel approach to uncertainty in risk – a generalized risk measurement through composition of belief revision operators (likelihoods L) that result in cascaded GTU-based operators (diversity and recusion of uncertainty types) given by:

$$\begin{aligned} &\pi_{\varphi(g_1,\ldots,g_k)} = \pi_{\varphi(g_1)} \circ \ldots \circ \pi_{\varphi(g_k)} : \phi \rightarrow \phi^u_{(g_1,\ldots,g_k)} \\ &R\left(B^u_g, \pi_{\varphi(g)}\right) = \mathrm{E}_g\left[\mathrm{L}\left(B^u_g, \pi_{\varphi(g)} \mid I_k\right)\right] \text{ (generalized risk operator)} \end{aligned} \tag{3.14}$$

based on the k-th stage information extraction (partial inception) I_k, updated belief system B^u_g, and the composite belief operator (action taken) $\pi_{\varphi(g)}$.

Time dilation models within the inception concept will be studied next. Each inception level dilates the relative time scale and hence makes possible computational speedup and so, should be considered time discounted recursive games because (i) each inception level is a new game module with input from the prior level and output to the next level, and has an option to jettison back to some original lower level (or reality level), and (ii) speedup of computation in each successive level can be translated to increased payoff increases in information, knowledge, and influence garnishment.

Very general physics-based forms of automata are investigated as models for recursive games in view of work that has shown an equivalence between recursive state pushdown automaton modular calls and two-agent recursive game winning strategies (Alur, La Torre, & Madhusudam, 2003; 2006). Inceptions are two-coalition games and can be considered as a coalition extension of two-agent games. We generalize inception models by utilizing those general automata models to be discussed and expanded on here.

The generality of the cumulative consciousness function q_i, will be considered when utilizing Zadeh's general theory of uncertainty (GTU) to express models. In the end, generalized uncertainty evolutionary models of automata will be considered as prototypes for inception games. Game payoffs , in these generalizations will be in the form of GTU-based consciousness functions. In this manner, a further generalization metamodel for inceptions will be GTU-based evolutionary recursive automata.

Inception Spacetime Models

According to the Inception concept movie, a time dilation/expansion occurs between dream levels, in addition to spacetime physics configurations not honored in prior levels. It was stated (in the movie) that an approximate 12 fold factor occurs descending from one level to the lower adjacent one. Denote a dream level by L_i where i signifies the dream level descension from a reality level of 0. Table 1 below displays the consistent movie dialogue implications and an exponential fit for time dilation/expansion between dream levels. Because of the uncertainty and ambiguity in the way the mind perceives hallucinatory and dream time, there will exist inconsistencies in logical time translations. Hence, these are merely approximate time translations using experiential evidence (from the character dialogue in the movie) and an exponential model for more consistent discussion (Proces, 2010).

Table 1 - Inception time dilation/expansion (in min with time relative to L_0 level)

Dream level	Time (according to Ariadne character, factor 12)	Time (according to Arthur character, factor 20 (closer to 22)	Time (using overall data from scenes)	Predicted time from exponential fit: *l* level, *t* time, $g(t,l) = (0.92293)te^{3.048l}$
0	1	1	1	.92
1	12	20 (22)	16.8	19
2	144	400 (484)	432	410
3	1728	8000 (10648)	8760	8637
4 (limbo)	20736	160000 (234254)	?	182008

Time expansion into a descended dream level presents with a marked advantage for the inception traveler because what-if scenarios can be simulated in dream level time frames while allowing for mere fractions of that dream time in level 0 reality. This can be viewed as a hyper-speed computational simulation in which modeling is achieved without laborious and time consuming data collection and analysis. Simulation-based modeling, such as can be done in generalized Monte Carlo simulations (Holmes, Gray, & Isbell, 2009), where probability distributions are generalized to distribute more possibilities, is a means to emulate inception what-if events. In the movie, limbo time is equated with L_4. However, here we will consider general levels, (i.e., the only true limbo is achieved by observing behavior at L_j as $j \to \infty$). Biological time becomes computational time in inception simulations for what-if scenarios.

Using a generalized exponential fit for computing absolute level times in L_l, given by $g(t,l) = ale^{bt} + k$, one may then approximate the inception movie time frame simulation. This then sufficed to solve certain types of prognostication in L_0. Compute time in L_0 remains tractable, while the corresponding compute time in L_l becomes intractable as $l \to \infty$. In our abstraction of inception, $g(t,l)$ is further generalized to fit certain hypercomputation models, most notably Zeno-type execution speedups where infinite numbers of computational steps are performed in finite times and hence, solving the halting problem in finite time as well (Weyl, 1949; Copeland, 2004). Gravity-relativistic time dilation according to the Schwarzchild solution to the Einstein field equations for relativity, utilizing the Schwarzchild metric, is expressed as

$$dt_E^2 = \left(1 - \frac{2GM_i}{r_i c^2}\right) dt_c^2 - \left(1 - \frac{2GM_i}{r_i c^2}\right)^{-1} \left(\frac{dx^2 + dy^2 + dz^2}{c^2}\right) \tag{4.1}$$

leading to the time dilation (contraction) ratio $\gamma_{(dx,dy,dz,dt_c),U}$,

$$\gamma_{(dx,dy,dz,dt_c),U} = \frac{dt_E}{dt_c} = \sqrt{1 - \frac{2U}{c^2} - \frac{v^2}{c^2} - \left(\frac{c^2}{2U} - 1\right)^{-1} \frac{v_r^2}{c^2}} \tag{4.2}$$

where dt_E is an increment of time as recorded by an internal frame, dt_E is an increment of time as measured by an independent clock's coordinate system, (dx, dy, dz) are corresponding spatial increments in the clock's system, $v^2 = \dfrac{dx^2 + dy^2 + dz^2}{dt_c^2}$ the coordinate velocity of the clock, v_r the radial velocity, and $U = \dfrac{GM_i}{r_i}$ is the sum of the local Newtonian gravitational potentials from masses with distances r_i from the measuring clock (Ashby, 2002). We thus thrust a more general notion of time dilation into inception levels using the inception level time function,

$$g(t,l) = \gamma^{-1}_{(dx_l, dy_l, dz_l, dt_{l,c}),U_l} t + k_{(dx_l, dy_l, dz_l, dt_{l,c}),U_l} \tag{4.3}$$

where for inception level *l*, the spacetime coordinate increments are $\left(dx_l, dy_l, dz_l, dt_{l,c}\right)$ with local Newtonian potential U_l and $k_{\left(dx_l, dy_l, dz_l, dt_{l,c}\right), U_l}$ is the sum of possible quantum fluctuation errors caused by the quantum foam perturbation of time with very small sub-Planck length interaction. Recent experiments have shown that the quantum foam perturbation effects may be at sub-Planck length scale ($< l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.61619997 \times 10^{-35} m$) and hence that spacetime is smoother than first thought by quantum physicists (Nemiroff, Connolly, Holmes, & Kostinski, 2012). Each inception level will then represent a succeeding level of very distant gravitational lensing, chained from the meta-reality of an original clock to further out levels of relativistic time-contracted regions. Inception levels are then, at least, separated by spacetime light cone horizons. In this framing, a concept of negative indexing of inception levels is introduced to mean hyper-reality or super-chains above the relative origin of consciousness of an agent of further time-dilated light cone horizons. Hyper-reality or hyper-inception negative levels are then equated with time contraction, while sub-reality is equated with higher positive inception levels of dilated time.

Notwithstanding recent developments and confirmation of middle-scale qubit (> 100-qubit arrays) QM computation in D-Wave machines utilizing quantum annealing algorithms in Boixoes, Albash, Spedalieri, Chancellor, and Lider (2013), technical speculation from researchers of near-future practical large-scale quantum computers utilizing the theory of closed timelike curves (CTCs) in general relativity and quantum gravity, may manifest the development of hypercomputational, near superluminal machines (Aaronson and Watrous, 2008). This class of machines may potentially provide answers to computationally hard questions (*NP*-complete as such) in the far future and then relay those answers back to the past, to the origin of those inquiries – hence time travel messages of solutions to the original computing mechanism in the past. To elude the grandparent paradox of backward time travel, (i.e., the paradox of traveling back to eliminate one's ancestors and having the capability to return to the current time without any path to conception), one constrains the region of spacetime where a causal consistency exists – a fixed point of some evolution operator in the evolution of the quantum state backwards, the Deutsch resolution (Deutsch, D. (1991). CTC-based automata (CTCAs) if CTCs exist and CTCAs can be built, were shown to solve PSPACE problems – problems needing polynomial amounts of memory with possibly EXPTIME time resources, using conventional circuitry – making space and time exchangeable as computational resources and hence as an alternative to quantum gravity computers that rely on SPACETIME resources. However, it was also shown that quantum CTCAs are no more efficient (powerful) than classical ones for solving PSPACE problems if CTCs can be shown to exist and then utilized to build such CTCAs. In this sense, if quantum gravity can be realized through the use of CTCs, classical machines would suffice for PSPACE problems which contain NP problems. We now briefly look at incremental computational differentiation between inception levels and what this may mean for speedup times.

Let $j > l$ and denote the compute time for a simulated event, *E*, in level *l*, using automaton *m*, by $C_m(E,l)$. Then

$$C_m\left(E,j\right) = g\left(C_m\left(E,l\right),j\right) = \gamma^{-1}_{\left(dx_j,dy_j,dz_j,dt_{t,j}\right),U_l} C_m\left(E,l\right) + k_{\left(dx_j,dy_j,dz_j,dt_{t,j}\right),U_t} \tag{4.4}$$

The time computation advantage, $\Delta(E,m,j,l)$, given an automaton *m* and event *E*, of entering level *j* from level *l* is

$$\begin{aligned}\Delta\left(E,m,j,l\right) &= C_m\left(E,j\right) - C_m\left(E,l\right) \\ &= \left(\gamma^{-1}_{\left(dx_j,dy_j,dz_j,dt_{t,j}\right),U_j} - 1\right) C_m\left(E,l\right) \\ &\quad + k_{\left(dx_j,dy_j,dz_j,dt_{t,j}\right),U_j}\end{aligned} \tag{4.5}$$

Hence, if an event *E* requires more computational time than is possible within the confines of a time deadline of t_l in its level L_l, then by entering into level L_j from level L_l, if

$$\Delta\left(E,m,j,l\right) \geq C_m\left(E,l\right) - t_l = \gamma^{-1}_{\left(dx_l,dy_l,dz_l,dt_{t,1}\right),U_l} t + k_{\left(dx_l,dy_l,dz_l,dt_{t,1}\right),U_l} - t_l \tag{4.6}$$

or using (4.5) and (4.4), if

$$C_m\left(E,l\right) \leq \frac{\left(t_l - k_{\left(dx_l,dy_l,dz_l,dt_{t,1}\right),U_l}\right)}{2 - \gamma^{-1}_{\left(dx_j,dy_j,dz_j,dt_{t,j}\right),U_j}} \tag{4.7}$$

the computation may be achieved for retrieval before the deadline expires in L_l. Let the class of all events, *E* in level L_l using machine *m* where this is true for some finite $j > l$ be denoted by $M(E,l,m)$. Then an inception agent in level L_l, with knowledge of machine *m*, wanting to simulate event *E* for purposes of information extraction, can do so eventually in some level $j > l$ provided $M(E,l,m) \neq \emptyset$.

In a discussion on time-discounted games (stochastic or deterministic) in Appendix B, discounts on future payoffs are given based on hyperbolic and subadditive discount factors. This discount scheme relates to the propensity of humans to underweight current period discounts compared to future period discounts and to the length of periods when payoffs are calculated (Read, 2001). In the general relativistic time computation resource advantage given by (4.5), time discounts can be given proportional to $\Delta(E,m,j,l)$–the computational resource differential between interception levels $L_j - L_l$. Assuming a simplified Markovian game (stationary and memoryless stochastic game) with infinite possible levels (stages) as a first model for inceptions, using (4.40) of Appendix B, one may write the cumulative payoff to an agent i, accounting for time discounting as proposed above going from inception level *j* to level *l*, utilizing the automaton *m* available at inception level *j*,

$$v^i_{k,r,m,l,j}(a,s)(t) \equiv \lim_{j \to l} \sum_{m=1}^{\infty} \frac{E_{(a,s)}\left[R_i(t)\right]}{1+k\left(\Delta(E,m,j,l)\right)^r} \tag{4.8}$$

Again see Appendix B, equation (4.40) for parameter details. However, inceptions are being modeled at face value as recursive games, a special kind of stochastic game where the only terminating plays are when one escapes levels with an inception (for the inception coalition). We will define our generalized version of an recursive game later in this section.

Evolutionary games in which action time through the evolution of introducing different (mutant) strategies from having more computational results, correlates directly with payoff, as in an inception game, will then possess possible dominant strategies if inception has been isolated to a closed group of co-opetive coalitions (Weibull, 1997). In that case, strategies invoked in the inception game must be evolutionarily stable (ES) for there to be asymptotically stable development in the inception. Coalitions in an inception game are equated with coercion teams that are sent into inception levels to extract information from other individuals or groups of interest to a paying client. We next investigate how emergent automata may emulate inceptions and their proposed decision structure as recursive games.

While these models for time dilation emulate inception level relativistic effects, spacetime-gravity physics are also modified. Gravity plays a large part in the alternative physics that transpires from one level to another. More appropriate spacetime models should include both large-scale gravitational and micro-scale quantum mechanical effects. To this end, we will consider proposed quantum-gravity (QG) models for inception levels, mostly through the lens of a generalized probabilistic approach named causaloids from Hardy (2007). Other models of QG will be considered in very brief passing such as Superstring (M-theoretic) and pre-geometric frameworks later in this study. We will now turn our attention to how computational models in the form of generalized automata may frame inceptions and how this will establish an equivalence utilized to establish reasonable game strategy types for inceptions.

Emergent Automata and Inception Games

Finite automata (finite state machines, FSMs) are given by a 5-tuple $\mathcal{A} = (Q, A, E, I, F)$ where Q is the set of states of the machine, A is the acceptable alphabet, $E \subset Q \times A \times Q$ is the set of transitions, and $I, F \subset Q$ are respectively the set of initial and final states. $\mathcal{A} \subset \Xi$ where Ξ is the space of all automata (with possibly infinite states and/or alphabets). Ulam and von Neumann developed the concept of cellular automata (CA), in a sense, generalizing finite automata by allowing for more flexible policies for changing states and by extending the reach of influence of a cell state to (possibly) non-local, finite *n*-dimensional generalized lattices of cells (Bianlynicki-Birula & Bialynicka-Birula, 2004). Each cell can be viewed as a FSM. A synchronized clock then counts time intervals and all cells change state according to the transition rules of the CA. Traditional CAs have localized causality, that is, cells are influenced by the state change of its topological neighbor FSMs. However, CAs may be generalized to include non-local causality according to some generalized transition rule which may include quantum entanglement or long-range relativistic effects in the case of quantum or quantum gravity-type CAs.

Formally, a cellular automata is given by the 4-tuple $\{Z_n, S, N, f\}$ where Z_n is an *n*-dimensional lattice (finite or infinite grid) of cells (each with FSM characteristics), S is a finite set of cell states, N is a topological neighborhood, and f is a transition function defined by a transition rule and acting on the neighborhood around each cell (Garcia-Morales, 2012). The global state of a CA is called a *configuration*, $c \in S^{Z_n}$ where

$|S| = k < \infty$. The position of a cell on the lattice is given by a position index $x \in Z_n$. In general, the multi-coordinate position index *x*, is ordered via a lattice ordering and a configuration *c*, can be expressed by this ordering through a string representation $c = \left(\ldots c_{x_{-1}} c_{x_0} c_{x_1} \ldots\right)$. If a finite subset of cells Z^* of the lattice Z_n are considered, a corresponding finite configuration for Z^* can be expressed similarly as $c^* = \left(c_{x_0^*} c_{x_1^*} \ldots c_{x_n^*}\right)$ where $x_i^*, i = 0,1,,,,n$ are the lattice ordering for cells in Z^*. The topological neighborhood *N*, defines the template for overlaying a neighborhood N_x, on a cell c_x using relative multi-coordinate positioning indices, $N = \left\{n_{y_i}\right\}_{i=0,1,\ldots,m}, m = |N|$ where the n_{y_i} are *n*-D coordinates describing the extent points of the neighborhood template *N*.

Applying a neighborhood template to a cell c_x, one obtains the cell neighborhood configuration, $n_{c_x} = c_{x+n_{y_1}} c_{x+n_{y2}} \ldots c_{x+n_{y_m}}$. The dynamic transition of cell c_x state at time *t*, denoted by c_x^t to the next time increment state c_x^{t+1} is given by $c_x^{t+1} = f\left(n_{c_x}\right)$, the local transition rule. The CA evolves based on the iteration $c_{n+1}^{t+1} = f\left(g\left(c_n^t\right)\right)$ where *g* defines the time-neighborhood to consider in the transitions, (i.e., $g\left(n^t\right) = \left(n^s\right)_{s \in N_t}$, for some set N_t that defines time indices around *t*). Here n^t is a time-neighborhood anchored around time *t*. The global transfer for this group-induced CA follows as before. The global transition function (rule), *F*, would then be expressed as the simultaneous application of the local transition rule to each cell at time *t*, $C^{t+1} = F\left(C^t\right)$ where

$F\left(\ldots c_{x-1}c_x c_{x+1}\ldots\right)=\left[\ldots f\left(n_{c_{x-1}}\right)f\left(n_{c_x}\right)f\left(n_{c_{x+1}}\right)\ldots\right]$. The transitions in CAs can be generalized based on Markov processes where memory extends past one time increment and further, on Zadeh GTU constraints. Consider a GTU constraint, G. Express the local transitions as $c_x^{t+1}=G\left(g\left(n_{c_x}^t\right)\right)$. The corresponding global transition rule G_G follows, $C^{t+1}=G_G\left(C^t\right)$ where

$$G_G\left(\ldots c_{x-1}^t c_x^t c_{x+1}^t\ldots\right)=\left[\ldots \mathrm{G}\left(g\left(n_{c_{x-1}}^t\right)\right)G\left(g\left(n_{c_x}^t\right)\right)G\left(g\left(n_{c_{x+1}}^t\right)\right)\ldots\right] \tag{4.9}$$

See Appendix B for details on Zadeh GTU constraints G. For finite CAs, transitions that overlap on boundaries can be wrapped around in a closed lattice mending at those boundaries or by using cyclical positioning via modulo operators. In a final generalization to CAs, one can consider the space of configurations associated with a CA, $C=A^H$ where H is a finitely generated group and A is an alphabet (Ceccherini-Silberstein & Coornaert, 2010). The group H defines much more general configurations than spatial sequences. The configuration space C is considered the phase space of evolutions for the CA expressed as a dynamical system (Capobianco, 2008). The lattice structure is provided by a Cayley graph of H. Then the CA can be equated with a continuous map, $f:C\rightarrow C$ which commutes with the group action of H. We will next review pushdown automata (PDA) and later consider the extension of CAs where each cell is a PDA, in order to produce a pushdown cellular automata (PDCA) in consideration of equivalent notions of winning strategies for PDAs and recursive games as inceptions will be viewed as recursive.

Pushdown automata simulate all traditional stack operated machines by using *pop* (delete), *push* (add), *nop* (empty) operators on a finite alphabet stack. They contained $\mathcal{A}$ in the sense of every FA has an equivalent PDA, but at least one PDA does not have an equivalent FA, but more importantly are advantageous over $\mathcal{A}$ because they can additionally store information in a stack as they process symbols. FSMs recognize exactly the regular (rational) languages – languages which are defined by regular expressions and are generated by regular grammars – whereas PDAs exactly recognize the context-free languages – languages with grammars that require less stringent requirements than regular grammars. In the context of games, languages, which are sets of strings of symbols from an alphabet, define the recognizable strategies that can be implemented. The winning strategy for a game is simply the language that is most recognizable, most efficiently. We define in more detail pushdown automata later along with a timed version when tying those automata with time dilated recursive inception level games. We investigate pushdown automata and systems which are simulated by recursive state machines (RSMs) since it has been shown that winning modular call strategies (guaranteed module call compatibility) are equivalent to winning strategies in recursive game graphs and hence to those in recursive games of which inceptions are timed versions thereof. It has been shown that pushdown automata can be approximated using RSMs (Alur, Etessami, & Yannakakis, 2001). RSMs and their modular processes or recursive subprogram calls under certain circumstances are equated with successful strategies in infinite state (stochastic) recursive games (graphs) if computation is interpreted as decision-making in Alur, La Torre, & Madhusudam (2003;2006) and are categorized in Fridman (2010). This approach to tying automata and their recursive subprogram calls to favorable stratagem in stochastic recursive game graphs where computations are decision branches is the theme to be used to expand on inception games as very general emergent automaton. Inception levels $L_l, l = 0,1,...$, are to be equated with recursive subprogram calls. In this light we consider emergent types of automaton and computing machines to equivalence to inception games. The recent concept of timed pushdown automata (TPDA) from Abdulla, Atig, & Stenman (2012) can then be utilized in relating inception level time dilation and game strategies with pushdown automata modular subprograms. These automaton models will then emulate generalized inception games using frameworks for general uncertainty and universal non-classical physical theories. These equivalences will put into view the powerful computational and game metaphors of inceptions. We start with hypercomputational automaton of the Zeno type, proceeding to quantum-gravity machines using general uncertainty frameworks and then to evolutionary, von Neumann, and Gödel automata leading to generalized pushdown automata where the favorable game stratagem equivalence to recursive modular subprograms is used.

Define a Zeno machine (ZTM) (accelerated Turing machine) in the following manner. A ZTM is a Turing machine with one input, output, and storage tape each. The *n-th* transition is computed in $\frac{1}{2^n}$ unit time. Theoretically, without being concise about defining and ascertaining a final state of an ZTM, infinite transitions (computations) or a supertask can be done within one unit of time. Potgieter (2006) suggests this represents a problem with ZTMs satisfying the Strong Church-Turing Thesis (SCTT) with respect to a ZTM framework but not to regular TMs. ZTMs serve as simplified models for hypercomputation and can therefore be theoretized about and applied to general hypercomputation Turing machines in general. Burgin (1984) and Putnam (1965) developed inductive Turing machines (ITM) in which a TM computes stable outputs after a finite number of computations and never changes thereafter. If a stable answer is found at the *n-th* transition, the ITM is said to be of order *n*. Though infinite time hypercomputational machines have been theorized and categorized by Hamkins (2005), we will want to concentrate on finite time hypercomputation since inceptions strategically require achievement in finite time periods with respect to level-0 reality, notwithstanding infinite time computations are approached perceptively at higher inception levels with respect to computations at lower inception levels as $l \to \infty$, (i.e. generalized inception limbo-time is infinite time computation).

Quantum-gravity, Hypercomputation, and Generalized Pushdown Automata for Inceptions

Time dilation computation is honored by near-*c* speed relativistic effects and hence, the movement of particles, fundamental to computation, whether classical, quantum, or quantum-relativistic (quantum gravity), follows appropriately in the various dream levels, L_l. Digital physics adheres to the position that any perceived reality is in the limit, at the discrete spectrum of Planck-level existence. Zizzi (2000, 2005a, 2005b) has proposed quantum gravity computation at the Planck level using digital loop quantum gravity (LQG) spinfoam models. ‘t Hooft (2012) proposes a (universal) physical cellular automata on 1+ 1 dimensions with Boolean processes acting on spacetime lattice sheets that map onto the operators of superstring field theory, hence a digital superstring QG based automata alternative to a LQG-based automata. More accessible proposals of quantum digital computation come from the sub-models of quantum finite and pushdown automata which have become well researched abstract models that recognize regular and non-deterministic languages – of which general semantic problems may be solved (Cem, Say & Yakaryilmaz, 2011; Golovkins, 2001).

Etesi & Nemeti (2002) establish models for hypercomputation in Malament-Hogarth spacetimes where an observer may witness infinite amounts of computation in finite time because of relativistic effects in proximity to black-hole horizons. Lloyd & Ng (2004) also consider machines based on black-hole relativistic horizon effects. Hardy (2007) has conceptualized the notion of a quantum-gravity automata using his causal-probabilistic framework for quantum-gravity referred to as causaloids. Lloyd (2006), Abrams & Lloyd (1998), and Czachor (1998) posit how a plausible quantum computer can simulate quantum gravity but is not intrinsically run by quantum gravity rules. Instead, small non-linear terms are added to the QM linear representations of state evolution, leading to capabilities to solve NP-complete problems efficiently, (i.e., in P-time). Recently, Kempf (2013) proposed the idea of utilizing EPR-type measurements via spectral geometric measurements on spatially separated entangled spin qubits in quantum vacuum to encode and simulate spacetime curvature in quantum computers and hence simulated a full QG computer, (i.e., the spectral geometric properties of spacetime curvature are akin to "hearing' what the shape of the spacetime curvature is). This suggests using more general entangled qubits, labeled as *e*-bits here, to emulate QG particles as transporters of information in a QG computer. In order to build a quantum gravity machine however, a more concrete theory of QG must emerge based on experimental agreement, successful prognostication of physical results such as those from particle discovery, and elegance of explanation as in theories of everything with clearer physical manifestations to computational theories. Why approach universal computation through the lens of emergent physical theories such as QG? While some natural physical processes may not be classically computable (e.g., Fouché (2000) showed that Brownian motion with respect to Wiener measure at rational points cannot be approximately computed by a classical TM in time almost surely); changing the definition of computability (via the various versions of the Church-Turing-Alonzo Thesis - CTAT) to encase non-classical phenomena is needed (Licata, 2012). New definitions of computability such as those emanating naturally from QG or evolutionary approaches must be formed and well-defined before positing that physical and computation processes are functionally isomorphic – the stronger Church-Turing-Deutsch Thesis (CTDT) (Deutsch, 1985). In this regard, QG theories are approached using either emergent or fundamental spacetime geometries and relativity and is spacetime geometry, if it exists, dynamical or fixed. Superstring theory is a quantum field theoretic approach that assumes an emergence of spacetime gravity after initially being fixed. LQG, spinfoams, etc. are background independent versions of QG that have emergent dynamical geometries (Markopoulou, 2009). Here we consider machines that handle generalized uncertainty due to the Zadeh GTU formalism and are quantum gravity-based automata that are capable of naturally computing quantum-relativistic events with integrated SPACETIME resources.

Inceptions, as introduced, are social phenomena. For those emergent casual physical models that are being considered here as general computational structures for inception games, this begs the relevance question, “where and how do inceptions take place outside of the social phenomena arena?” However, inceptions can theoretically form at the quantum (and near-Planck levels) and cosmic levels through general subatomic-quark particle and galactic clusters-dark energy-dark matter (gravitational lensing) interactions respectively.

Causaloids Λ, are systems of extended conditional probabilities that endeavor to connect well-defined probabilities of events interpreted by programs, F to compute in fuzzy spacetime regions, R – regions in which local (fuzzy) spacetime causality is bounded – connecting these regions back together causally via compression operators on three successive complexity levels of composition. Hardy refers to fuzzy spacetime regions as those in which space-like or time-like qualities are unknown apriori because of indefinite causal and temporal structures. We may generalize Hardy’s notion of regions by imposing a general stochasticity to its space-like or time-like features. A Zadeh uncertainty structure, G may be imposed on region R and be denoted as R_G. See Appendix B for more details on Zadeh’s general theory of uncertainty (GTU) and its tuple constraint form *G*. Causation is therefore developed in GTU quantum-gravity as general (time) indefinite causal structures.

Formally, causaloid quantum gravity computers (CQGC) are given by pairs, $Q = \{\Lambda, S\}$ where Λ is a causaloid system and *S* is a set of gates (state evolvers). See Figure 3 in Appendix C for a rendition of a lattice CQGC. Along each edge *e*, computational particles, which could be qubits or *e*-bits, but may be further generalized to be their GTU counterparts that we label as *g*-bits in Sepulveda (2011), travel and encounter at node *l*, decision evolvers which is a decision branching process to divert the particle to a subset of possible gates, $S_l \subset S$. Note that in a CQGC, the gates S_l may be partially non-local to node *l* (in the classical topology of the lattice). *Q* has to be a non-time step computer since it is dictated by an indefinite temporal structure. In a Hardy CQGC, probes labeled by an index $n \in N_p$ are omnipresent in QG spacetime. In a Loop QG formalism of QG, these probes would reside at spinfoam network nodes, the elemental structures of LQG (Rovelli, 2008). In a superstring (M-theory) version of QG, the probes would be represented as primitive string configurations. Hardy probes, p_n, $n \in N_p$, in the vernacular, can be Planck level mass ($l_p^3 = 4.224x10^{-105} m^3$) computational machines with systematic inputs x_n (spacetime reference point), program structure F_n and systematic output a_n that may be the detection of physical manifestations of qudits (multi-level qubits) such as groups of photons. The inputs are governed by the execution of the program, F_n on the input x_n and on a spacetime physical rendition of history data, r_n (time-independent) received by the probe, so that $s_n = F_{p_n}(x_n, r_n)$. Data is thus collected as probe information tuples (cards) $q_n = (x_n, a_n, \mathrm{s}_n, F_n, r_n)$. Let $V = \{q(x,a,s,F,r)\}_{x,a,F,r}$ denote the set of all possible cards. A subcollection of such, $V_{N,I} = \{q_{n_j}\}_{n_j \in N, j \in I}$, $N \subset N_p$ where *I* denotes an index of repeated runs of probe calculations and *N* is a subset of probes from N_p, will represent the stack of cards developed from the experiment (N, I) of runs on a subset of the probe space N_p. Further, let $R_x = \{q(x,a,s,F,r) \mid x\}_{(a,s,F,r)}$, denote the subset of cards with input *x*, $Y_s = \{q(x,a,s,F,r) \mid s\}_{(a,\mathrm{x},F,r)}$, the set of cards with output *s*, and $V_F = \{q(x,a,s,F,r) \mid F\}_{(x,a,s,r)}$, the subset of cards consistent with the program *F*. Finally, one defines a composite spacetime region, $R_O = \bigcup_{x \in O} R_x$. The outcome set of R_x is depicted as $Y_{R_x} = \{Y_s\}_s \cap R_x$ and the program in R_x is depicted as $F_{R_x} = \{V_F\}_F \cap R_x$ In order to stochastically connect one quantum spacetime region to another, conditional probabilities (or GTU constraint operators, G_p as proxies or extensions of classical probability) of the form,

$$G_p(Y_{R_x} \mid F_{R_x}, Y_{R_z}, F_{R_z}) \tag{4.10}$$

must be well-defined and calculable. The uncertainty operator in (4.10) will be well defined if the regions are casually connected. To make this plausible, the uncertainty values (and operators), $G_p\left(Y_{R_x} \mid F_{R_x}\right)$ must be well defined. To this end, Hardy imposes an assumption on the outputs and programs in the complement (in *V*) of a very large region $R \subset V$, $\frac{card(V \setminus R)}{card(R)} \sim 0$. Conditions *C*, are put on $F_{V \setminus R}$ and $Y_{V \setminus R}$ such that the values $G_p(Y_R \mid F_R)$ are well defined. In a quantum gravity computer, the large region *R* is where the computations function. Otherwise, computations are nonfunctional or meaningless.

Denote by $\mu_x = \left(Y_{R_x}, F_{R_x}\right)$ the measurement pair depicting both the measured output and the measuring apparatus program on spacetime regions. The GTU operators of the form,

$$G_{\mu_x} = G\left(Y_{\mu_x} \cup Y_{R \setminus R_x} \mid F_{\mu_x} \cup F_{R \setminus R_x}\right) \tag{4.11}$$

are then considered as the components of states, expressed as minimal expanses (fiducial sets), Ω_1,

$$G(R_x) = \begin{pmatrix} \cdot \\ \cdot \\ G_{l_1} \\ \cdot \\ \cdot \end{pmatrix}, \; l_1 \in \Omega_1 \tag{4.12}$$

of the region R_x where linear relationships between the probabilities (4.11) and the states are posited via the same physical logic utilized for quantum linear representations,

$$G_{\mu_x} = G_{\mu_x}(R_x)G(R_x) \tag{4.13}$$

where $G_{\mu_x}(R_x)$ are scalars dependent on the region R_x. The matrices, Ω_1 are then considered compressions of these states. Causaloids next consider two more levels of compression via state representations in higher order matrices. See Appendix C for details on the compression series that is manifested by a causaloid system. In the case of a spacetime lattice with pairwise interacting particles, $\left(l_{x_i}, l_{x_j}\right)$, regions R_x, and adjacent particle pairs $\left(k_{x_i}, k_{x_j}\right)$ intercepting at nodes x. Causaloid system are expressible as, $\Lambda = \left(\left\{\Lambda_{\mu_x}^{l_{x_i} l_{x_j}}, \forall x\right\}, \left\{\Lambda_{l_{x_i} l_{x_i}'}^{k_{x_i} k_{x_i}'}, \forall \text{ adjacent } x, x'\right\}\right)$ subject to $\Lambda_{l_{x_i} l_{x_i}'}^{k_{x_i} k_{x_i}'}$ calculation methods (clumping) and to how the nodes in the spacetime lattice that place hold theoretical qubit or generalized particles, dynamically act or move – the so-called causaloid diagram. Causaloid diagrams are akin to Feynman diagrams for lattice node elements in this spacetime regionally linked causal framework. Appendix C reviews this calculus for building causaloids under a conventional Kolmogorovian probability framework, whereas here we consider more general GTU operators, (i.e., operators which may be general expressions of combinations of fuzziness, uncertainty such as rough set approximations, possibilities, internal quantum-gravity probability, paraconsistent logics, etc.).

Consider causaloid computer classes as follows: a class of causaloid computers is the set $C_\Lambda^m = \left\{\{\Lambda, S\} \,\middle|\, |S| \le m\right\}$ for some integer *m*, where *S* is a subset of gates, $S \subset S_\Omega$, S_Ω is the universe of spacetime lattice gates, and Λ is a causaloid system. Hardy considered *m* to be small on the order of 10 for realizable computer operation spaces. However, here we do not put a restriction on *m* and indeed consider infinite dimensional causaloids as we do inception levels. A causaloid computer instance $\left\{\Lambda, S_m^U\right\}$ in C_Λ^m is *universal in* C_Λ^m if $\left\{\Lambda, S_m^U\right\}$ can simulate any computer instance in C_Λ^m. Hardy established that causaloids exist that simulate both classical and quantum computers via causal definite structures and the probability structures that connect them. In a QG causaloid computer no such definite causal structure exist nor may classical probability structures connect them, even with quantum probabilistic rules since relativistic effects are not accounted for in the connection. QG causaloids are then built based on QG theories such as LQG, Superstring theories, their respective derivatives and flavors or so-called pre-geometric approaches (Caravelli, 2012). For example, in LQG spinfoams are used to connect 3+1D spacetime regions such that quantum and relativistic effects are honored in a background independent manner. In an LQG version of a causaloid, the probabilistic structures are replaced by spinfoam paths. In our GTU version of causaloids, probabilities are replaced by generalized constraints *G*, via Zadeh's definition. GTU causaloids with QG spacetime regions would then generalize physical QG theory-based causaloids. We denote a GTU causaloid by Λ_G. The class of computers, $C_m^{\Lambda_G}$, *m* an integer, represents regional QG causaloid computers for small *m*. Universal causaloid computers, $\left\{\Lambda_G, S_m^U\right\}$ in $C_m^{\Lambda_G}$ would define an equivalence class in the space of all causaloid computers $\mathcal{G} = \left\{\Lambda, S\right\}$ based on Turing equivalence. However, even universal causaloid computers may not be able to honor the CTDT. Here, we simply theorize GTU causaloid computers with input, output, programs, and with storage requirements (tape) fulfilled by the collective state of the contained traveling particles (*g*-bits), a GTU causaloid automata. We now investigate placing internal clocks at operational instances of nodes in order to simulate timed pushdown automata.

Time pushdown automata (TPDA) are PDAs in which clocks (infinitely many possibly) are placed at stack nodes and the value of those times constrain the actions of the automata (Abdulla, Atig, & Stenman, 2012). Stack ages are also compiled and hence the SPACE requirements of TPDAs are larger than those of PDAs by an order of the number of internal clocks placed. To this end, recall that a pushdown automata is a tuple (S, s_0, A, Δ) consisting of states *S*, an initial state s_0, *A* is a stack alphabet of possible acceptable symbols, and Δ is a set of transition rules. Pushdown operators include adding or removing information to the stack – push and pulls respectively. Transition rules are of the form (s, op, t) where *s* is the source state, *t* is the target state and *op* is the operation, *push(a), pop(a)* or *nop(a)* where $a \in A$ is an arbitrary symbol and *nop(a)* is the *empty* operation of not modifying the stack while changing the state *a*. Reachability of an automata is the capability of deciding whether it can reach a state *s*.

TPDAs are PDA tuples with clocks, (S, s_0, A, Δ, C) where *C* is the set of clocks. In TPDAs, two additional operators are added based on clocks, $x \in T_I$?, (i.e., is the value of clock *x* in the time interval T_I?) and $x \leftarrow T_I$, (i.e., deterministically resets the value of clock *x* to some value in the time interval T_I). Stack symbol ages are stored as well as the clock values, *x*. For purposes of studying reachability in TPDAs, the concept of regions (from timed automata) are modified to fit the role of region equivalence in TPDAs. Regions in a TPDA are subsets of the automata where clock valuations (times) are similar modulo some arithmetic rule. Clock values *x* are separated into their integral part *INT(x)*, and fractional part, *FRAC(x)*, respectively. Two configurations are equivalent if for the configuration clock valuations, *x* (i) *INT(x)* are the same within a threshold constant c_{max}, (ii) $FRAC(x) = 0$ or $FRAC(x) > 0$ in both, and (iii) the order of the values $FRAC(x)$ are identical in both. Time sequences are then calculated by rotating values, (i) if $FRAC(x) = 0$ for some items then *push()* is instigated, or (ii) if $FRAC(x) > 0$ for all items then the items with largest $FRAC(x)$ values are incremented, (i.e., $x \rightarrow INT(x) + \delta(x)$). Next, two types of items are defined, plain items and shadow items. Plain items represent clock values or stack symbol ages. Plain items consist of the set $X \cup A \cup \left\{ \quad \right\}$ where $\; = 0$ except when *pop()* is executed. Shadow items record the values of corresponding plain items in the region below and consist of the symbols $\left\{ x^*, a^*, \; ^* \right\}$ and are recorded to remember the amount of time that elapses while the plain items they represent are not the topmost stack items. The timed transactions, $x \in T_{I?}, x \leftarrow T_I, pop(a, T_I)$ are also simulated by PDAs operations. See Abdulla, Atig, & Stenmen (2012) for more details.

Timed pushdown GTU causaloid automata (TPDGTUAs) can be represented using the tuples of TPDAs, with a GTU causaloid, Λ_G , $\left(S, s_0, A, \Delta_G, C, G \right)$ and utilizing the stochastic transition rules of GTU causaloids to execute the modified stack operations

$\Delta_G = \left\{ pop_G(a,T_I), push_G(a,T_I), nop_G(a,T_I), \left(x \leftarrow T_I \right)_G, \left(x \in T_I\, ? \right)_G, {}_G \right\}$ appropriately. For example, $pop_G(a,T_I)$ is executed according to the GTU rules (4.13) and the time interval T_I is replaced by a corresponding spacetime region R_I. C is a set of clocks positioned at the causaloid spacetime lattice nodes and is replaced by spacetime measurement probes indexed by n as discussed earlier leading to the definition of causaloids. These changes are necessary because the concept of time in a TPDGTUA is merged into spacetime since general indefinite causal structures are involved and hence timed sequences are non-sequitar. Hence, the timed operations are replaced by spacetime operations utilizing the GTU causaloid structure making the possibility of inter-regional computation well-defined.

We return to the discussion of recursive state machines (RSMs) and their role in pushdown systems that contain submodular automata. In Alur, La Torre, & Madhusudan (2003) modular strategies for infinite recursive games modeled by recursive graphs are equated with RSMs and recursive procedure calls that model the control of sequential processing. Favorable or winning strategies for recursive games are equivalent to a $\omega-$regular specification (infinite strings formed from A as a regular language) over the observables and is modular in the sense that resolution of choices within a module does not depend on the context in which the module is invoked, but only on the history within the current call. These problems of modular reachability (games) were shown to be EXPTIME-complete in general and N_p-complete for fixed sized modules. Alternating temporal logic (ATL) is utilized to show that the delivery of a message in a module A is independent of the behavior of another module *B* in a recursion using winning recursive game strategies governed by the $\omega-\text{regular}$ specification.

That winning modular strategies are computable for RSMs and recursive game graphs implies that memoryless strategies implemented in each module lead to compatible recursive calls. In the context of game representations, recursive game graphs can be mapped onto their equivalent class of recursive games in normal form, albeit with some loss of information not affecting strategy outcomes. However, in light of how inceptions can be viewed as recursive games, winning strategies can be had without retaining or taking full advantage of the histories of other inception levels or modular calls. In inception, the notion of subcoalitions and *n*-agent strategies complicates the run of the game where various types of uncertainty of information reigns supreme. Risk spectra become flattened out because of these uncertainties. By the simplification of guaranteeing the compatibility of each inception level transfer or module call by the use of the equivalent $\omega-\text{regular}$ specification translation in inception game information, winning or achieving inception (or avoiding inception for the opposing inceptee group) can be deterministically arrived at. It remains to show what module call (inception level transfer) compatibility and $\omega-\text{regularity}()$ mean in the context of multi-agent coalition inception games.

Pushdown automata can be further generalized based on the rules of transition and the actual apparatus physics. In this way the concept of quantum pushdown automata (QPDA), along with quantum finite automata (QFA) were introduced in Moore &

Crutchfield (1997) and corrected in Golovkins (2001). We review the definition of a QPDA,

Def. A quantum pushdown automaton (QPDA), given by the tuple, $A = (Q, \Sigma, T, q_0, Q_0, Q_r, \delta)$ where Q is a finite set of states, Σ is a finite input alphabet, T is a stack alphabet, $q_0 \in Q_0$ an initial state, Q_a and Q_r are mutually exclusive accepting and rejecting states respectively in Q, and $\delta : Q \times \Gamma \times \Delta \times Q \times \{\downarrow, \rightarrow\} \times \Delta^* \rightarrow C[0,1]$, where $\Gamma = \Sigma \bigcup \{\#, \$\}$ is an input tape alphabet of A and $\{\#, \$\}$ are end-markers not in Σ . $\Delta = T \bigcup \{Z_0\}$ is the working stack alphabet of A and $Z_0 \notin T$ is the stack base symbol, $\{\downarrow, \rightarrow\}$ are direction operators of the input tape head.

Well-formedness conditions, including separabilities, are put on these components in order to stay faithful to the conditions of quantum mechanics. See Golovkins (2001) for these details. We now define the configuration for the quantum pushdown automata,

Def. A configuration of a quantum pushdown automaton is a pair $|c\rangle = |v_i q_j v_k, \omega_l\rangle$ where the automaton is in state $q_j \in Q$, $v_i v_k \in \{\#, \$, \Sigma^*\}$ is a finite word on the input tape,and $\omega_l \in Z_0 T^*$ is a finite word on the stack tape. The input tape head is above the first symbol of v_k and the stack head is above the last symbol of ω_l. We denote the set of all configurations of a pushdown automaton by $C = \{|c\rangle\}$.

In the l_2-normed Hilbert space generated by C, $H_A = l_2(C)$, any global state of A can be expressed in the basis $\langle |ca\rangle\rangle$ as: $|\psi\rangle = \sum_{c \in C} \alpha_c |c\rangle$ where the sum of the squared amplitudes are unity, $\sum_{c \in C} |\alpha_c|^2 = 1, \alpha_c \in \mathbb{C}$. Linear operators U_A on the states of A are expressed as

$$U_A |\psi\rangle = \sum_{c \in C} \alpha_c U_A |c\rangle \tag{4.14}$$

For a general configuration $c = |v_i q_j \sigma v_k, \omega_l \tau\rangle$,

$$U_A |\psi\rangle = \sum_{(q,d,\omega) \in Q \times \{\downarrow, \rightarrow\} \times \Delta^*} \delta(q_j, \sigma, \tau, q, d, \omega) |f(|c\rangle, d, q), \omega_l \omega\rangle \tag{4.15}$$

where $f(|v_i q_j \sigma v_k, \omega_l \tau\rangle, d, q) = \begin{cases} v_i q \sigma v_k, \text{ if } d = \downarrow \\ v_i \sigma q v_k, \text{ if } d = \rightarrow \end{cases}$. The separability conditions imposed by Golovkins (2001) implies the well-formedness of QPDA. Language recognition for QPDAs are then defined in the following manner:

For a QPDA $A = (Q, \Sigma, T, q_0, Q_0, Q_r, \delta)$, define three configuration subsets, $C_a = \{|v_i q v_k, \omega_l\rangle \in C \mid q \in Q_a\}$, $Q_r = \{|v_i q v_k, \omega_l\rangle \in C \mid q \in Q_r\}$, and $C_n = C \setminus (C_a \cup C_r)$. Let E_a, E_r, E_n be the subspaces of H_A spanned by C_a, C_r, C_n respectively. Denote by *o*, the observable that corresponds to the orthogonal decomposition $H_A = E_a \oplus E_r \oplus E_n$.

Measurement of *o* will then either be “accept”, “reject”, or “non-halting”. Now consider a word $x \in \Sigma^*$. This is input into *A* as the string $\# x\$$. Assume that the initial configuration in *A* is then $|q_0 \# x\$, Z_0\rangle$. In a first step the operator U_A is applied to the current global state of *A*, leading to a second step of the resultant superposition observed using the observable *o* defined above. Assume that the global state of *A* before the observation *o* is $\sum_{c \in C} \alpha_c |c\rangle$. The probability that the superposition is projected into E_i is therefore $\sum_{c \in C_i} |\alpha_c|^2$ for $i \in \{a, r, n\}$. The computation continues until the resultant observation is “accept” or “reject”. Using these mechanisms of language recognition, a QPDA is shown to recognize every regular (rational) language. Other irregular languages such as $L_3 = \left\{ \omega \in (a,b,c)^* \,\middle|\, |\omega_a| = |\omega_b| = |\omega_c| \right\}$ are recognizable by a QPDA with probability $\frac{2}{3}$ and $L_5 = \left\{ \omega \in (a,b,c)^* \,\middle|\, |\omega_a| = |\omega_b| \;\; xor \;\; |\omega_a| = |\omega_c| \right\}$ with probability $\frac{4}{7}$ respectfully, where $|w|_i$ denotes the number of occurrences of the symbol *i* in the word ω.

Cellular automata can be extended to have pushdown automata capabilities by forming pushdown cellular automata (PDCA) where each cell is now a PDA (Kutrib, 1999). Each PDA cell with state at time *t*, c_x^t, receives a collective input (string) from its designated neighborhood group of cells, $g(n^t) = (n^s)_{s \in N_t}$ with content $c_x^{t+1} = f\left(g\left(c_x^t\right)\right)$. It then processes the input utilizing its stack (transition) operations

$\left(\mathrm{c}_x^{t+1}, {}_i op_x^{t+1}, c_x^{t+1}\left({}_i op_x^{t+1}\right)\right)$ where ${}_i op_x^{t+1}$ is the *i-th* executed operation and $c_x^{t+1}\left({}_i op_x^{t+1}\right)$ is the resultant output state. Let

$$
{}_f c_x^{t+1} = \begin{pmatrix} \left(\left(\mathrm{c}_x^{t+1}, {}_1 op_x^{t+1}, c_x^{t+1}\left({}_1 op_x^{t+1}\right)\right), {}_2 op_x^{t+1}, c_x^{t+1}\left({}_2 op_x^{t+1}\right)\right) \\ \ldots\left(\mathrm{c}_x^{t+1}, {}_{f-1} op_x^{t+1}, c_x^{t+1}\left({}_{f-1} op_x^{t+1}\right)\right), {}_f op_x^{t+1}, c_x^{t+1}\left({}_f op_x^{t+1}\right)\right) \end{pmatrix} \tag{4.16}
$$

denote the final output at the time *t+1* increment for cell c_x where ${}_f op_x^{t+1}$ designates the final operation executed for that cell at that time increment. Since these calculations are in general not in lock-step, synchronization issues appear. Using the last-out parallelization synchronization policy, the outputs, ${}_f op_x^{t+1}$ where the cell c_x is in the neighborhood group, $g\left(n^t\right) = \left(n^s\right)_{s \in N_t}$ surrounding another cell, c_y, contribute accordingly to that cell's initial input state at the next time increment ${}_1 c_y^{t+2} = f\left(g\left(c_x^{t+1}\right)\right)$. PDCAs are special, albeit general lattice cases of automata arrays which are important metamodels for parallel computation. CAs which are associated with groups whose alphabets are affine algebraic sets, generalize conventional lattice-based CAs. Further to this, discrete geometric-topological objects described by metric topologies or algebraic-topological objects with very general algebraic structures on semi-lattices (affine, relational, Lie, Heyting, Clifford, or division algebras) further abstract the shape and extend of cellular automata working in tandem according to some regime of interaction rules and neighborhood types. Ceccherini-Silberstein & Coornaert (2010, 2011) develop the case for affine algebraic cellular automata.

Von Neumann machines/automata (VNA) are versions of automata arrays that self-replicate to form macro-automata from micro-automata (cells) within the scale of the automata array (Kutrib, 2001). VNAs can therefore be further generalized by considering more abstract shape distributions of micro-automata in the macro-automata through more exotic topological and algebraic structures. For example, a topological lattice may be non-primitive in the sense that the lattice curves around itself and has more than one lattice point per cell at a spatial location. We will discuss in more detail von Neumann automata later in this section. CAs may be opportunistic models for inception games because inception levels can be viewed as evolvers much like the iterative mappings in CAs and multi-agent coalition groups can be delineated amongst cell populations. The evolution of CAs may define the dynamic of inception coalition teams as inceptions are attempted navigating through different inception levels. To this potential, evolutionary dynamics of simple Boolean CAs has been heavily studied since the initial conception of von Neumann (1951,1966) up to the recent attempt to classify CAs into four classes and 256 prototypes by Wolfram (1983, 2002), with a further analytic definition of a universal map for Boolean CA generation in Garcia-Morales (2012). The evolutionary dynamics of more general algebraic-topological CAs has not been systematically approached.

Sepulveda (2011) stipulates how a generalized uncertainty automata for information processing can be developed using a recurring thematic tool in this study; Zadeh's notion of a general theory of uncertainty (GTU) that subsumes and categorizes through logical constraint and score functions, uncertainty models including: classical Kolmogorov and Bayesian probabilities, quantum-probability, fuzzy and rough set probabilistic and evidential theories, possibility theory, Dempster-Shafer type theories of evidence, and other anthropomorphic notions of risk and uncertainty (Zadeh, 2006). These automaton are referred to as GTU Turing machines or automata (GTUTAs). Causaloid-based quantum gravity uncertainty from Hardy (2005) is framed into this patternization as well. See Appendix B for a detailed discussion on the GTU.

Finally, the author combines a notion of morphogenic computation utilizing generalized objects as computational units replacing the familiar Boolean algebra with mathematical fields using its topoi (generalized sets from category theory) equivalent as the standard unit of computation – morphogenic Turing machines (MORPHTMs). In a further attempt to generalize automata and information fields, the author posits about the development of evolutionary automata in the framework of MORPHTMs. MORPHTMs are given evolutionary recursive operations of variation (crossover) μ, and selection ρ to produce evolutionary MORPHTMs or EMORPHTMs.

As a natural motivation for approaching decision making from an evolutionary point of view, Cooper (2003) posited that decision theory formed from decision tree analysis is an extension of a more general evolutionary tree analysis whose decision branch plans are referred to as life-history strategies. Cooper seats evolutionary theory as the root generator of all logics, including decision theoretic logics, inductive and deductive logics, and mathematics. Brenner (2008) goes further and grounds many functional logics such as quantum logic as special cases of an overarching non-propositional logic based in relational ontological physical reality named the logic in reality (LIR). Apart from the major work developed from evolutionary game dynamics utilizing stochastic differential forms as discussed in Appendix B, one can approach game dynamics using evolutionary processes from equivalent automaton models. In this spirit, recently Burgin (2013) defined generic evolutionary automata as sequences of TMs that act as one-input two-output progeny machines. Burgin refers to such automata as general evolutionary *K*-machines (*K*-GEMs), where *K* is a class of automata with one input and two outputs.

Def. A *K-GEM* is a sequence $E = \{A_t\}_{t\in T}$ of automata from *K* that operates on population generations, $\{X_i\}_{i\in I}$ which are coded as words in the alphabet of *K*. The objective of the *K-GEM* is to conceive of a population *Z* that satisfies:

1. A_t (level automaton) of *E* represents a one-level evolutionary algorithm operating on the input generation X_i, applying recursive variation and selection operators, μ and ρ respectively.
2. X_0 is the initial generation and is operated on by A_1 consequently generating subsequent generations, X_1 (transfer output) that inputs to A_2.
3. A_t receives input from either A_{t+1} or A_{t-1}, then applies the operators μ and ρ to X_i producing X_{i+1} as its transfer output and when necessary (non-terminating node) sends it to either A_{t-1} or A_{t-1}.
4. The optimal search condition to select a population agent x_{i*} from X_i is $x_{i*} = \arg\max_{x_i \in X_i} f(x_i)$ for some fitness performance measure *f* (Burgin, 2013).

Burgin signifies that a *K-GEM* is inductive of order *n* if each of its members is at most inductive of order *n*. Burgin further defines universal evolutionary automata (U) in much the same way that universal TMs are via codification.

Def. Let *H* be a class of evolutionary automata. An evolutionary automaton/algorithm/machine *U* is *universal for H* if given a coding $c(A)$ of automaton/algorithm *A* from *H* and input data *x*, *U* obtains the same result as *A* for input *x* or gives no result when *A* gives no result. An evolutionary automaton/algorithm/machine *U* is *universal in H* if it is universal for *H* and $U \in H$ (Burgin, 2013).

In a further abstraction to evolutionary processes, Sepulveda (2011) defines a generalized evolutionary process by injecting generalized uncertainty into the selection of a class of fitness functions at each stage *t* given by a Zadeh GTU structure, $g = (X, r, R, ts)$ where *X* is a constraint variable, *R* is a constraining relation, *r* is an index representing the modality of constraint semantics (uncertainty model such as possibilistic, intuitionistic, probabilistic, quantum-probabilistic, fuzzy, rough set theoretic, etc.), and *ts* is a test score functional associated with the GTU constraint (Zadeh, 2006). We define g_t to be the GUT operator that selects the class of functions F_E to be considered for a optima at stage *t* from a universe of fitness function classes, $\mathcal{F}$ for the evolutionary automata *E*.

Denote a GTU universal evolutionary Turing machine (GUETM) by $\Upsilon = (E, \mathcal{F}, g_t)$ where E is a universal evolutionary Turing machine, $\mathcal{F}$ a superclass of classes of fitness functions, and g_t a GTU (constraint) object. EMORPHTMs or morphogenetic evolutionary automata consider replacing word (utilizing bit representations) construction with field construction. From this perspective an EMORPHTM, *G*, considers classes of fields such as those from quantum-gravity field theories (loop quantum gravity spin foam general networks and string fields from superstring theory) and dynamicism within them defined by uncertainty operators of quantum operators captured by an appropriate GTU object. The relativistic nature of a quantum-gravity can be simulated by the recursive nature of time dilation in each inception level if inception games can be structured as kinds of EMORPHTMs. We a will later pursue the nature of the connection between general automata and general recursive games since it will be our premise that inceptions are general recursive games. This will tie inceptions into the structure of generalized dynamic automata that can be viewed as equivalent to recursive games in terms of logical information control flow. Next, however, we consider even more general adaptive evolutionary machines.

If $E = \{A_t\}_{t \in T}$ is an evolutionary automata, its components, A_t are apriori defined then actionized by the dynamics of the optimization of the fitness function *f*. Indeed, the components of *E* may be dynamically born instead of being given a prior existence. Consider automata that are capable of two progenic operations, self-reproduction and self-improvement. von Neumann (1966) famously developed a class of self-reproductive automata that defines most research in self-building machines. A more contemporary definition of von Neumann machines follows.

Def. *Kinematic self-replicating von Neumann automata/machines* are quadruplets, $SR = (\phi, A, B, C)$, where ϕ is a set of blueprints (code instructions) that completely describe how to construct the triplet of machines, *(A,B,C), A* is a constructor automata/machine capable of building another copy of *SR* given ϕ, *B* is a blueprint copier automata/machine, and *C* is a controller automata/machine that synchronizes the control of alternating actions of *A* and *B* (Freitas & Merkle, 2005).

Alur, La Torre, & Madhusudam (2003; 2006) and Walukiewicz (2001) showed a computational equivalence between pushdown automata as recursive state machines (RSMs) and two-agent recursive games (graphs). Moreover, effective strategies in two-agent recursive game were equated with modular control flow in recursive procedure calls, (i.e., the component recursion games in two-agent recursion games are independent local memory modular recursive procedure calls). RSMs consist of component machines called modules that each have a set of nodes representing internal states and a system box that contains entry and exit nodes with edges connecting nodes and boxes. Edges entering a box are invocations of the module represented by that box, while edges leaving a box are the returned values of that module. In a RSM two-agent game, the set of nodes is divided into two partitions representing each agent. The agent strategies are then the transition rules for invocation from their respective nodes, (i.e., the edges chosen to proceed from their nodes). We define recursive game graphs as in Alur, La Torre, & Madhusudam (2003;2006) in a finite automaton context and as a way to connect to the concept of more general extensive form recursive games of which inceptions will further generalize.

Def. A (finite) recursive game graph *G*, is a series of modular game graphs, $G=\{G_i\}_{i=1,\dots,n}$ where $G_i=\left(N_i,B_i,\{V_i^k\}_{k=1,\dots,\mathrm{M}},Y_i,En_i,Ex_i,\delta_i\right)$ are modular flat game graphs, and for each modular game, N_i are the nodes, $En_i\subseteq N$ are the entry nodes, $Ex_i\subseteq N_i$ are the exit nodes, B_i are the boxes, V_i^k are the mutually exclusive set of places (nodes and boxes) occupiable in G_i by agent *k*, $Y_i:B_i\to\{1,\dots,n\}$ are assignments of boxes to a game module, and if $\mathrm{Calls}_i=\{(b,e)\,|\,b\in B_i,e\in En_j,\ j=Y_i(b)\}$ is the set of calls in G_i and $\mathrm{Retns}_i=\{(b,x)\,|\,b\in B_i,x\in Ex_j,\ j\in Y_i(b)\}$ is the set of returns in G_i, then $\delta_i:N_i\bigcup\mathrm{Retns}_i\to 2^{N_i\bigcup\mathrm{Calls}_i}$ is a transition function that governs the possible movements from nodes and returns from one game to nodes and calls of another game. Conditions are then put on *G* so that an automaton program can be emulated. A *play* in *G* is a path in the game graph. The *global state* of a recursive game graph $G=\{G_i\}_{i=1,\dots,n}$ is a pair (θ,u) where $\theta=\{b_i\}_{i\in 1,\dots,r}\subseteq B$ is a sequence of boxes (stack of recursive calls) and $u\in N$ is a node (current control point). Denote Q_G to be the space of well-formed states for *G*.

Since the set of places (nodes and boxes) can be partitioned by those belonging to *K* coalitions, a global state (θ, u) can be labeled as a *K*-state if either *u* is not an exit node and is a *K*-node (a node populated by coalition *K* agents only) or if (b_r, u) is a *K*-return (a return node populated by coalition K agents only). $Q_K \subseteq Q_G$ will denote the set of *K*-states. A global game graph corresponding to *G* is given as the tuple $GL_G = \left(Q_G, \{Q_k\}_{k=1,\ldots,K}, \delta\right)$ where there are *K* coalitions and δ is a global transition function $\delta : 2^B \times N \to Q_G$ under certain move, call and return conditions Reachability to certain nodes called *target nodes* from a starting node is considered a *winning strategy*. In this respect, nodes in a winning strategy are connected by eventual reachability. Using this definition, a *recursive reachability game* is a tuple $G = \left(\{G_i\}_{i=1,\ldots,n}, e_1, T\right)$ where e_1 is the entry node of G_1 and $T \subseteq Ex_1$ is the target set of target nodes. A winning global strategy in a recursive game is a winning strategy in the corresponding global recursive game. Finally, it was shown that recursive games as defined here are game graph isomorphic-equivalent to pushdown automaton and that by considering modular strategies defined on global memoryless, but fully local memory game modules instead of a global strategy, that winning modular strategies exist when those strategies are hierarchical in structure. Hierarchical strategies in a reachability game are those strategies (one per game) that make no recursive calls. Formally, a *hierarchical strategy f* on (G, e_1, T) is one in which no play following the rules of *f* (according to *f*) and starting from e_1 reaches a state of the form $s = \left(\{b_i\}_{i=1,\ldots,i}, u\right)$ where $u \in N_l$ and $b_j \in B_l$ for some $l \in \{1,\ldots,\mathrm{n}\}$ and $j \in \{1,\ldots,i\}$. Two computational complexity results were also shown for recursive reachability games of this sort. First, the problem of checking whether a coalition (or agent) wins a game (has a winning strategy) is N_p. Second, the recursive game reachability problem is N_p-complete. If the modular strategies are not hierarchical, the problem of reachability is undecidable.

If an inception game can be expressed as a recursive reachability game, the target set may be given by those nodes that express an inception state (i.e., the inception team has reached guru-consciousness or social power status as defined before). Nonetheless, inception games are highly non-hierarchical in general since inception-hoping is bi-directional.

In inception games, two coalitions are presented by the inception and inceptee teams and transact as two-agent games when the coalitions are stably cohesive. Previously, in our introduction of inception games, definitions were given for a strongly social stability. Recall the cumulative consciousness aware probability functions, $q_i = \sum_{l \in S \setminus \{i\}} \sum_{j=1,2.,,,m_i} p_{lj}(k)$ assigned to an agent i in an inception. The individual consciousness aware probabilities, $p_{lj}(k)$ may be replaced with well defined GTU constraint functions, $G_{lj}(k)$ assigned by a GTU constraint variable $g_{lj}(k)$ and test score function t_{ljk} for each triplet (l, j, k). This GTU can represent general uncertainty models and hence generalizes the representation of the inception payoffs and ensuing optimal strategy formation when defining Nash equilibrium and evolutionarily stable strategies. The strongly social stability of the inception game is predicated by these GTU constraints.

Recursion in Inceptions

Inceptions may also be viewed as psycho-social recursive games. The recursion structure occurs at each inception level decision point where the coalition teams battle for (non)entry into level *i* inception with a probability $p_i\left(a_I, a_{I^c}\right)$ given that the team-collective actions $\left(a_I, a_{I^c}\right)$ are executed by the two teams respectively. Here $a_I = \left(a_I^i\right)_{i\in I}$ and $a_{I^c} = \left(a_{I^c}^i\right)_{i\in I^c}$ are vectors of actions of each team member in the inception and inceptee teams respectively. Recall that a classical recursive game, $\Gamma = \left\{\left\{\Gamma_i\right\}_{i\in L}, H, A, S, p, p_\Gamma\right\}$ consist of (1) two agents (teams), (2) a finite set of *n* games, $\left\{\Gamma_i\right\}_{i\in\{1,2,\ldots,n\}}$, (3) an action space, A (not necessarily finite), (4) a finite state space, S, (5) transitional probabilities, $p\left(z' \mid z, a\right)$ of going from state *z* to *z'* given the two-agent action vector $a \in A$, (6) an absorbing game probability of $p_\Gamma\left(a\right)$, given action vector $a \in A$ and (7) a vector of generalized payoffs $H = \left(H_i\right)_{i\in I_s}$ where $H_k\left(s_k^1, s_k^2, \Gamma\right) = p_k e_k + \sum_{j=1}^{n} q^{kj}\Gamma_j$, e_k is the payoff if the *k* strategy $\left(s_k^1, s_k^2\right)$ of the two teams is used, and the probability conditions hold, $p_k, q^{kj} \geq 0$, $p_k + \sum_{j=1}^{n} q^{kj} = 1$. For each game, Γ_i, either a terminal state is reached or entry into a next game, Γ_k is executed (Everett, 1957). In inceptions, an inception level *i* is equated with a game Γ_i in the whole inception recursive game, $\Gamma = \left\{ \left\{ \Gamma_i \right\}_{i \in L}, H, A, S, p, p_\Gamma \right\}$ with the inception and inceptee teams acting as the respective agents. Actions in inceptions are the allowable operations performable by each team such as deception, persuasion, brain-washing (thought insertion), termination, coercion, self-interest, team cohesion, and multi-leveled covertness (generalized double, triple, …*n*-switching agents). The states of the inception are the collective state descriptions of each team's status within the game environment, (i.e., inception (not) achieved, levels of partial inception, inception reversal, team termination, game termination). The state transitional probabilities and inception entry probabilities are self explanatory and can, in instance of uncertainty categories, be iteratively conditioned on updated Bayesian priors, (i.e., subjective Bayesian probabilities based on inception evolution and histories).

With this analogue between inceptions and recursive games, one must note that inceptions are more general than recursive games in the following manner. Inceptions may be iterative in the sense that inception teams may exit inception levels, ascending and descending into and out of other inception levels during epochs of zigzag inter-inception level activity. Successful inceptions are ultimately described by thought inception of one team at the hands of the other and of the successful ascension to level-0 reality of both team head operatives. In recursive games, once an exit is made from an intermediate game, two options are possible: (i) an exit from the global game or (ii) entrance into the next level game. Inceptions can instead, proceed: (i) to exit (ascensions up to level-0), (ii) to the next descension level game (inception), or (iii) to an intermediate level game (inception) through a series of inception transitions. Coalition (inception/inceptee) teams can be split into separate inception levels, working in time synchronization via the time rules discussed earlier, (i.e., agents in a coalition may reside in different levels during a stage or game status). These recursive games are labeled as multi-level, multi-stage coalition recursive games. The state of an inception game is dependent on a coalition definition of agent game states in possibly different game inception levels. Each inception game level must then retain its state with respect to all agents from both coalitions. When an agent i_K from coalition team *K*, re-enters a game Γ_L from a lower level game Γ_{L+j} (possibly via repeated or chained ascensions), the game state of Γ_L has changed based on the other agent's potential behaviors (s_{-i}) between the time of exit and the re-entry of that agent (with possibly more of an informed strategy space) into Γ_L.

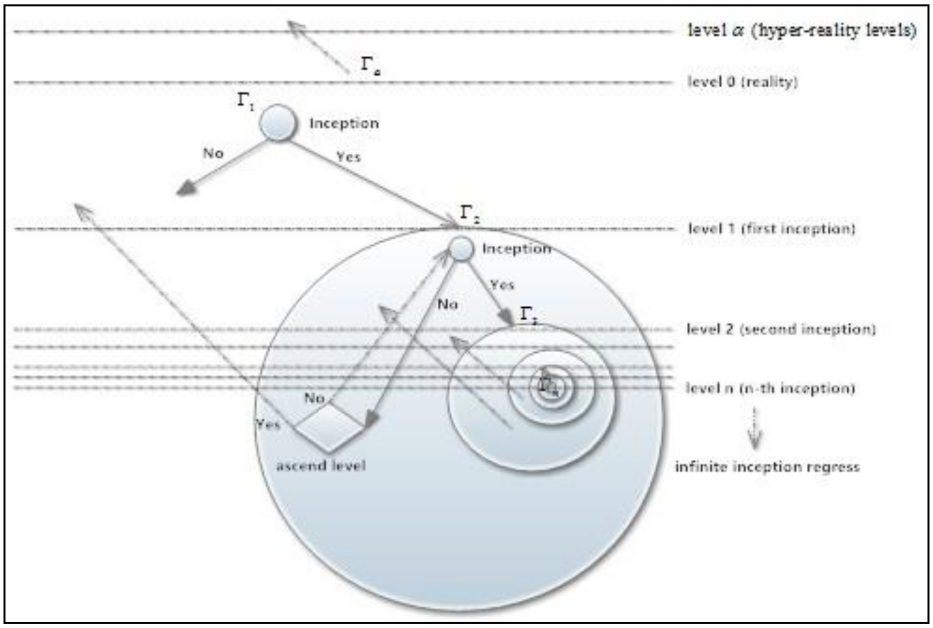

Figure 6 - Inception as generalized recursive game

In our generalized version of inception, levels can ascend past level 0 reality to hyper-reality or enlightened levels with ascending negative indices. Operative advantages in these hyper levels work analogously, (i.e., hyper-reality inception acts as an existential ascension from other lower levels and the more unstable reality level 0). Moreover, the payoffs given for recursive games are generalized for GTU constraints,

$H_k\left(s_k^1, s_k^2, \Gamma\right) = G_k e_k + \sum_{j=1}^{n} G_{kj}\Gamma_j$, for GTU functions G_k and G_{kj} generalizing the probability structures p_k and q_{kj} respectively, producing the GTU inception game

$\Gamma = \left\{\left\{\Gamma_i\right\}_{i\in L}, H, A, S, G, G_\Gamma\right\}$, where G and G_Γ are GTU functions generalizing p and p_Γ, the probabilities for state transitions and absorbing into a recursive game module (inception level existence) respectively. Stochastic recursion and time discounting should be accounted for because each inception level requires resources for time-accelerated computation. To that end, a time-discounted version of recursive games in inceptions can be expressed using a generalized subadditive discounting (See Appendix B for definition of the time-discounted game value $\left[{}_{s}v_{\beta}^{i}(t)\right]_{i=1,2}$ and the corresponding strategy, a^{*}),

$$H\left(s_{k}^{1}, s_{k}^{2}, \left[{}_{s}v_{\beta}^{i}(t)\right]_{i=1,2}\right) = G_{k}e_{k} + \sum_{j=1}^{n} G_{kj}\, {}_{s}v_{\beta}^{i}(t) \tag{4.17}$$

Virtualization and Visualization of Risk in Inception Games

The effective visualization and eventual virtualization of real time risk measurements in inception games is the final proposal we will investigate in this study. In our prior discussion on virtualization, Peircean virtualization is, at best, a 4th order approximation to reality. In the case of measuring risk in a general game structure, normally a payoff measurement is taken at the end of each game run in which agent stratagem are executed and corresponding payoffs are calculated. Traditional risk to an agent before a strategy is executed is measured as the expected utility of the agent given the strategies and resultant returned values (loss or gain) to that agent. Subjective measures of expected utility were reviewed in Appendix A, including those in emergent situations such as quantum probabilistic and causaloid settings, culminating in a general GTU-based risk operator. Virtualization nonetheless, refers to both approximations to reality and proactive and interacting feedback mechanisms, mostly through skin pressure-based haptic and tactile stimuli. Here we will greatly generalize these feedback spaces through multi-dimensional and sensorial apparatus and visualization, specifically tuned to risk spectra measurement – risk is a multi-dimensional object on a general space of possibilities and combinations of psychological and analytic profiling of the DM.

Risk may be generalized to measure not only the relative utility – the inner product of uncertainty measures of choosing a stratagem with the value (payoff) output of the game from executing those stratagem – but also the amount of irrationality apriori to the decision point with respect to the gamble situation – the loss and gain magnitudes – and the risk spectral (risk aversion to risk aggressive behavior) psychological point of view of the DM. This is at the least, a 3-dimensional object representing generalized risk of a DM at a specific decision point or branch of a game. Additionally, the evolution of risk measurements and decision making should be visualized (and virtualized through just-in-time and ahead-of-time what-ifs measures). We propose a simultaneous measurement of these risk components, including magnitude of loss(gain) and differentials of risk (instantaneous risk change rates).

Humans react more effectively to visual cues than numerical ones and hence dramatic and dynamic visualization for risk will uniquely improve a DM's arsenal of tools (Lima, 2001). Traditional proposals have been given and used for expressing the visualization of decision making , including color and 1-D size changes of geometric objects such as variations of linear and circular dials and the use of static geographic map data, heat maps, etc. (Roth, 2012; Bowman, Elmqvist, & Jankun-Kelley, 2012). Bordley (2002) proposes using decision rings instead of decision (or game) trees to illustrate consequences of decision paths. In Bordley-type decision rings, annular ring sections in a circle represents levels of decisions, specific events are represented as segments within those rings sections, relative sizes of those sections are probabilities of those events happening, payoffs are represented as coloring in those segments, and expected utilities are calculated using mixed colors from outer segments assigned to color inner segments (Bordley, 2002). Bordley's goal is to visually comfort the DM with a non-analytic qualitative uni-object representation of decision-making. The rings are 2-D and hence are limited in their spectral prowess. Klein (2002) proposed a taxonomy of visualization types depending on dimensionality and context type where multi-dimensional information types are best displayed and conveyed by the use of stacked visuals such as parallel coordinates – polygonal lines which intersect horizontal dimension axes at the various positions that correspond to the singlet dimensional value of a multi-dimensional value point.

Revisiting the concept of risk atoms introduced in the literature review, there is, nonetheless, a proper context in which to place micromorts – the underlying probability distributions that define the odds of a loss in a micromort and the estimation of their defining parameters based on population samples. Denote a microprobability of event *E* with an estimated pdf of h_S based on a population *P* by $\mu h_S PE$. In this case, the pdf h_S will also envelope into it the statistical estimator (operator), *S*, that defines h_S based on *P*. Besides the infinitesimal nature of a 10^{-6} probability event, there is no magical definition attached to micromorts. More generally, smaller units of event risk may be formed so that a new dynamic atom of event risk can be defined as $\zeta h_S PE$ where ζ is a negative integer power of ten, $10^{-\zeta}$. In itself each atom $\zeta h_S PE$ does not convey a value of loss or gain. For that to be expressed, the event E must be assigned a relative value $\nu_E^{Q,T}$ that depends on an estimate (using a statistical operator T) of a consensus from the population Q. The true ubiquitous and relative risk atom is thus defined as $\zeta h_S PE\nu_E^{Q,T}$. With this parameterization, a rational DM would pay $\zeta h_S PE\nu_E^{Q,T}$ for the microprobability $\zeta h_S PE$.

In a stochastic (or GTU constraint-based) process, s_t, over a period of time $[t_k, t_l]$, a measure of risk through the number of risk units $\zeta h_S PE\nu_E^{Q,T}$, may be given in terms of the underlying (period) distributions, event, populations, and statistical operator(s) given in $\zeta h_S PE\nu_E^{Q,T}$. In a game-theoretic framing, an action a (or strategy profile π) has consequences at each stage and over a period of time as well. Hence, a DM can measure risk relative to a strategy profile π in a game G, by assigning a number of appropriately formed risk atoms $\zeta h_S PE\nu_E^{Q,T}$ with the underlying parameters being defined by the game construct. Finally, in this study one replaces probabilities h with GTU constraints g, to generalize the uncertainty situation for a game and hence, for risk atoms, $\zeta g_S PE\nu_E^{Q,T}$, where S is (are) uncertainty operators or algorithms that define the estimates of g_S.

The visualizations we envisage are based on multi-dimensional representations of the various aspects of risk, as proposed before, including risk psychological profiling of the DM, displayed as floating 4-D polytopes (ellipsoids in the prototype) that collide with each other in plane intersections. Ethical sensorial extensions to 4-D spacetime visual limitations of humans are made via various sensorial stimuli including chromatic, olfactory, skin-pressurization, thermoception, wavelength optics, surrounding air movements, auditory spectra, proprioception (body awareness), bariception, equilibrioception (balance), pilorection (hair follicle extension during stress), micro senses, intuitive fuzzy senses as functions of physical senses, and emergent synergistic metasenses from combinations of individual senses. Some studies have suggested a range of reasonably delineated human senses between 14 and 31, many being functions of the five main sensory systems of sight, hearing, taste, smell, and touch. Nociception (pain) and interoception (hunger) are two subjective, but possibly uncomfortable receptor senses that will not be included as well as those receptor senses that tend to mask or confound one another's measurements or highly speculative psychic or telepathic senses. As important as the stimuli that are sensed are the memory systems that store the remembrance of those sense values – sensory memory which consists of autonomic systems of iconic (visual), echoic (hearing), and haptic (touch) memory types (Colttheart, 1980; Dubrowski, Carnahan, & Shih, 2009; Claude, Woods, & Knight, 1998). While these autonomic-based sensory memory systems are limited in time, there resources are important to an overall time computational solution for DMs to more optimally process risk analytics via these simultaneously mapped sensorial interpretations. The question remains, with a newly expanded resource of sensory risk evaluators, what is the working memory and cognitive load for this new extrasensorial risk analytics? Additionally, psychophysics endeavors to find answers pertaining to the thresholds of detection and differentiation in sensory systems (i.e., subliminal levels of stimuli and threshold boundary probabilities) (Gescheider, 1997). Detection thresholds will signal the presence of a stimuli and differentiation threshold will convey the smallest differential between levels of stimuli that is detectable by the subject (Peirce, & Jastrow, 1885). These sensory thresholds therefore delineate the possible invariant measurement parameters for the risk-to-sensorium mapping so that if $\rho_{\det}$ and ρ_{diff} denote the detection and differentiation thresholds respectively, then an affine risk-to-sensorium mapping $\beta_k : \Pi_r \to s_k$ that maps the *k-th* risk component of the game risk tuple Π_r, to the assigned sensory system s_k is scaled invariantly by $\beta_k\left(\pi_0 + \lambda\pi\right) = \beta\left(\pi_0\right) + \lambda\beta\left(\pi\right) = \rho_{\det} + \rho_{\text{diff}}\, s_k$, where $\rho_{\det} = \beta\left(\pi_0\right), \rho_{\text{diff}} = \lambda$ are the lowest sensory stimuli point and stimuli unit mapped from the lowest risk value and risk unit respectively.

In sensory systems, sensory receptor cells and neurons are mapped uniquely by sensory receptor molecules with distinguished rhodopsin molecules that possess idiosyncratic chemical reactions to external feedback which can both inhibit other receptors while intensifying their own with the exception of certain color optical receptors in larvae during metamorphoses (Mazzoni, Desplan, & Celik, 2004) (Hofmeyer & Treisman, 2008). Pathways to sense receptor neurons or nerve endings (for non-neuronal receptor cells), along with those specific receptor cells represent unique sensory patterns in sensory systems. Each sensory pattern can then be mapped to a potential sensory system. In this way, psychosomatic potentials for unique mammalian senses are numerous, far surpassing the posited range of 14 to 31 human senses mentioned before.

Including the possibility of metamorphoses of sensory systems through lifecycle changes in organisms, the true range of unique senses becomes dynamic with neuronal and nerve receptor reconstruction. Hence a theoretical limit to the number of potential senses in mammals is only limited by the mortality of the robust subject. With this potential for a human sensorium, information stimuli can be directly mapped to unique sensory nerve or neuronal receptor systems. In a recent experiment with remote noninvasive EEGs, a brain signal corresponding to a right hand push down button movement from a sender subject was successfully received by a receiver subject transmitted over an Internet connection (Rao and Stocco, 2013). This example points to future refinements of brain signal transmission of more subtle communications.

Nerve or neuronal receptor stimuli are triggered by chemical thresholds to be matched with the relative risk component value, activating gradient potentials which in turn initiate transduction in cells (see Figure 3). In this manner, a risk measurement is conveyed directly to a particular sensorial system in the DM sensorium. Here we use the word *sensorium* to refer to the universe of potential sensory systems in an adaptive living organization As an example thermoreceptors may be divided into spectrally regional receptor systems that have adapted during the lifetime of the organism, (i.e., separate hot and cold spectra reception via transient receptor potential – TRP proteins) (Viana, la Peña, & Belmonte, 2002). Within these thermal spectral regions, components of risk such as risk aggression/aversion may be mapped as succinct chemical stimuli to the appropriate thermal sensory regions. Recently, Someya (2013) has developed, in lab prototypes, a bionic skin interface that is capable of sensitizing external stimuli to touch neurological centers, which succinctly digitizes those simuli. Such interfaces labeled *e-skins*, can then connect and map external measurements of risk spectra to appropriately calibrated neurological stimuli, enhancing traditional means of visual information display.

In a potential cyclic causal feedback mechanism, the increase in risk aggressiveness has been postulated to be linked to increased dopaminergic activity, especially in adolescent puberty. Post adolescent self-regulation tendencies shift the pendulum back to risk aversiveness (Steinberg, 2008). Increased levels of the bio-chemical neurotransmitter dopamine actuates increased risk taking and hence a tendency to certain decision strategies rewarding large gains in the midst of large potential losses. The cause-effect cycle of visualizing a sensory system that is correlated to increased dopamine blood-brain barrier amounts in assessing a risk component may then be recursive in itself in a risk sensorium measurement.

Natural sensing of risk, using the evolution of virtual hair-on-the-back-of-your neck moments, can help develop map isomorphisms between components of a risk manifold $\mathfrak{R}$ and the human sensorium manifold $\mathfrak{H}$. Traditional quantitative graphics depict singular components of risk, usually in 1, 2, or 3 dimensional plots. On needs to expand this sensorium to include a functional subset of that full sensorium. We will consider in the next section a DIY holodeck-tyep approach in depicting this functional subset of $\mathfrak{H}$.

Spacetime-quantum components can be mapped to more visually effect communication of risk components. For example, consider the tuple

$\Pi = (\rho, P, q, G[S, g, p, Q])$ consisting of ρ – risk aggression spectrum, P – payoff magnitude, q – quantum-gravity probability frames (or GTU generalized constraints), $G_n[S_n, g_n, p_n, Q_n]$ – evolution of recursive game components (S_n, g_n, p_n, Q_n) ,where for the *nth* evolution, S_n is the state space, g_n is the GTU constraint, p_n is the agent space, and Q_n is the payoff functional. The visual geometric object representing components of this tuple will be called an *i*-morph (information morph). This object is capable of morphing (or being morphed by an external stimuli such as user touch, poke, or lathing) due to the

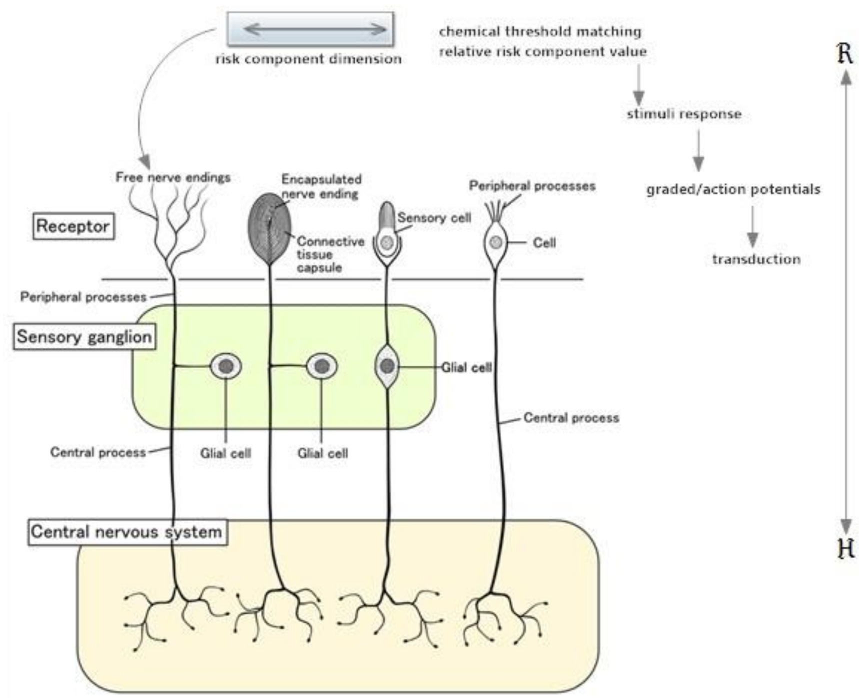

Figure 7 - Mapping risk components to sensory systems

dynamics of the risk space within the evolution of games (in particular, the recursive nature of inception games). Each orthogonal dimension of the i-morph represents the value of a risk component. The width, length, and depth of a geometric 4-D *i*-morph object relative to its centroid will be mapped to the tuple Π with the time component mapped to the evolutions $G_n\left[S_n, g_n, p_n, Q_n\right]$ in the following manner: as time t progresses, the game components are mapped to recursion game *i*-morphs that are in turn, visually connected to the original *i*-morph representing the three risk components.

Aside from the interaction of *i*-morphs which will be described later, a generalized divergence metric, D will be applied to two or more inception games in order to measure global similarities, (i.e., how similar are inceptions to each other in terms of risk and game components including equilibria spaces). See Appendix B for a discussion on divergences as generalized distance metrics in general spaces. See Figure 4 for a visual description of *i*-morphs. In that figure i-morphs are visually portrayed as 3-D ellipsoids, one for each spatial dimension, along a time dimension. However, in general, an *i*-morph can be a general 4-D object with sophisticated shapes projected onto each coordinate plane and a time evolution that may have curvature based on relativistic effects. Quantum effects are portrayed as fuzzy boundaries on the geometric *i*-morph object. In Porikli & Yilmaz (2012), silhouettes, general skeletal figures, and object contours are proposed to represent non-rigid objects for video analytic tracking. Corresponding general non-rigid *i*-morphs may also be represented by these generalized shapes, expanding to general stick figure Delaunay triangulations, 4-D silhouettes and object contours that can be changed in multi-faceted mathematical transformations pertaining to risk components.

These generalized *i*-morph objects represent the visual geometric sensory stimuli for conveying risk information to the DM. The general risk object will be sensed through a compendium of sensory systems – the risk sensorium. Visual geometric objects and evolution of corresponding game components within an inception game are mapped at the active inception levels (local automata), globally, at a bird's eye view of inception risk given by a measure of convergence to an inception and the qualifying risks that are part of that epoch in the inception game, and at mesoscopic levels which are at the boundaries of local inception levels and global inception, the spacetime where decisions are transcended to enter or leave inception levels.

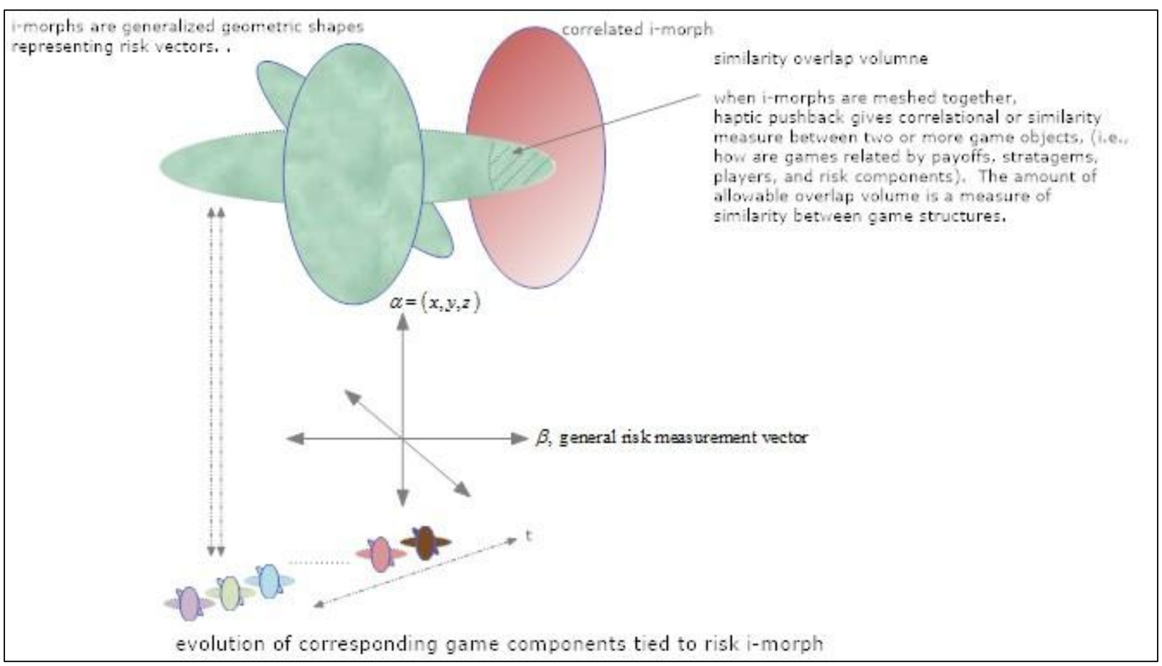

Figure 8 - *i*-morphs representing risk information vector

Interaction is proposed in which moving, nudging, poking, touching, or lathing *i*-morphs is a means to collide (or manipulate individually) two *i*-morphs in order to measure their similarities and dissimilarities. Mid-air displays of information objects as depicted in (Rakkolainen, Hollerer, DiVerdi, & Olwal (2009) in the form of *i*-morphs will be prototyped for optimal navigational flexibility in 4-D holodeck space. The spans of the overlapped region and the feedback force against pushing one against another will give relative impressions of similarity. Touch against pushing and pulling objects and interactively moving those objects in heads up display (HUDs)-like spaces will build an environment of immersive measurement of multiple risk scenarios and inception games.

In order to receive information conveyed as a risk vector in multi-sensorial manners, an accommodating environment must be utilized. This environment must meet the goals of multi-sensorial and dimensional presentation, interaction, and simulation. Immersion has been shown to lead to more productive processes in the military over the period of display technologies used (Hooper, 2004). Such an environment is approximated best via a VR world such as a Star Trek-style holodeck. Holodeck construction consists of 3-D 360^{o} surround immersive fully embodied simulation of visual, perceptual, skin pressure sensitization, body mechanical interaction, and other sensorial interaction. While true holography consists of the generation of overlapped images using diffraction grating and light interference, currently, autostereoscopic techniques utilizing binocular perception of 3-D depth, are mostly employed to simulate 3-D images for holographic imaging especially with 3-D non-glassware technologies. The early spectrum of holographic methods, techniques, and technologies were investigated in Hooper (2000). Currently, the most effective and practical holographic methodology is multiscopy in autosteroscopic images (Markov, Kupiec, Zakhor, Hooper, & Saini, 2006). Multiscopic techniques for autostereoscopy employ multiple camera perspectives including the most recent developments for a low power version, HR3D (Lanman, Hirsch, Kim, & Raskar, 2010). Holodecks are quickly converging to the virtuality part of the Milgram spectrum of mixed reality visual displays (Milgram & Kishino, 1994). Cave automatic virtual environment (CAVE) environments approximate these immersive simulations based on enclosed or nearly enclosed room-sized cube areas where projections on the inner walls (or subset thereof) of a CAVE are given that simulate 3-D world movements and visualization. The CAVE notation also alludes to Plato's allegory of the cave–reality as shadows on cave walls in his Republic (Allen, 2006).

The authenticity of Star Trek-style holodecks relies on 3-D holographic projection and matter reconstruction. CAVEs do not utilize holographic images. Holography is a mathematical method of reconstructing a 3-D image with Gabor-transforms of 1-D wavefronts overlapped in such a way as to add component fidelity as whole images (Gabor, 1948;1949). Matter reconstruction while theoretically possible in cellular 3-D printer reconstruction based on accurate biological substrates, is not near-future practical. On contrast, modern digital holography has made progressive strives to the point of the development of fast frame, near video rate 3-D holographic streaming video (holovideo) with modest computing resources and minimally lagged multi-level (touchable pressure-level sensitive) haptic holographic systems (St.-Hillaire, Lucente, Sutter, et al, 1995;

Plesniak, Pappu, & Benton, 2003; Page, 2013; Redux Sound and Touch, 2013). In Takayuki, Tatezono, Hoshi, & Shinoda (2008) and Takayuki, Tatezono, & Shinoda (2008) ultrasound waves are used to simulate tactile and haptic touch on holographic images. Haptic holography is important to manipulate risk *i*-morphs in our scheme because multi-level pressurization sensitivity will convey the DM's more precise desire to change or evolve decision risk and strategies for proceeding with inception games. Each risk component is able to be manipulated based on strategy rules. Those manipulations may mean directional change as well as magnitude. Hence moving and changing the shape of *i*-morphs will be equated with this multi-faceted manipulation of risk components.

First and early second generation holodeck prototypes are built using separate components that simulate visual, haptic, body movement interaction and other sensory stimuli such as auditory and olfactory. Although a system has not been proposed or built using more than 5 sensorial stimuli, we propose that a multi-sensorial holodeck can be built using commercially available components for each aspect of a holodeck experience. An example of a practical framework for building a prototypical holodeck is given by Project Holodeck using the Oculus Rift head mounted 3-D visor video feedback set, a Playstation Move optical system for motion tracking (upgrade to Playstation 4 and Eye Camera for our DIY simulated holodeck), a Razer Hydra system for body tracking, and the Unity3D game engine software system (Project Holodeck, 2013). The Oculus Rift HMD immersion is based on a field of view (FOV) of 110^{o} diagonal and 90^{o} horizontal, creating a spherical mapping image for each eye with a resolution of 1920x1080 and 6

DOF low-latency tracking. Oculus Rift has a low cost SDK. Dioptric correction is manual on the headset and interpupillary adjustments are made via software.

The Microsoft Xbox One Kinect 2.0 HD system may be adapted to fit into this holodeck schema replacing the Playstation Move. Triangulating (coordinating) three (or more) kinect systems will make more premise tracking components and panoramic viewing for this 3-D framework. The Kinect 2.0 system utilizes a light ray radar mechanism to measure macro movements at a few feet of distance from the sensor. The proposed DIY holodeck will envelope this triangulation scheme for tracking. Active moving is not addressed in this system. Hyperkin has developed a prototype system using the Xbox One Kinect 2.0 with an Oculus Rift HMD executing the action role-playing open world (non-linear) video game Skyrim (World News Inc, 2013).

Notwithstanding these technologies, we investigate more general and powerful natural user interfaces (NUI) – the ability for humans to interact with computers, providing inputs at the level of human sensorium such as micro-tracking finger, hand, arm, leg, speech and facial gesturing and touch pressurization sensitivity among other natural human output conduits (Wigdor and Wixon, 2011). These developments are part of the zeitgeist of the *disappearing UI* theme (Jasti, 2013). The recent explosion in NUI technologies and more efficient computational algorithms for converting location coordinates has given rise to the realization of micron-level tracking precision through the use of Leap Motion's gesture-based 3-D motion sensing (and capturing) proprietary system developed by D. Holtz (Foster, 2013). This sensor utilizes algorithms that measure differentials in light shadings at object boundaries as those objects move.

Virtual haptic interfacing is also possible through NUIs through continuous tracking of light hues and shades. The use of mathematical shape analysis is also making more possible fast near-zero delay computational times for object shape morphisms that are at the heart of novel NUI software mechanics (Sinha and Ramani, 2012).

Figure 9 - DIY simulated holodeck components

The use of micron-precision 3-D motion sensing interfacing with *n*-D holodeck displays, is the ultimate goal of the risk holodeck system. In this sense, this proposal for an *n*-D (risk) holodeck coupled with a logical extension of 3-D motion sensing systems to *n*-D interfaces, is a generalization to even Star Trek-like holodecks. An *n*-D motion sensing system would approximate a continuous tracking of every human nuance such as micro-expressions, body, muscle (through electrical activity), and eye movements, and other natural human output streams such as speech language idioms and idiosyncrasies. The Peircean differential between cognitive and real activity is further blurred by an *n*-D interface, *n*-D holodeck display and a final ingredient of an *n*-D simulators in the form of extensions to 3-D printer technologies, chemical emulators, and the novel approach of building pure von Neumann and Gödel simulators as automaton that self-propagate at the physico-computational levels.

The inclusion of a Virtuix Omni treadmill adds omni-directional walking surfaces for simulated natural user movement. The Razer hydra system may also be replaced with an ARAIG vest that simulates multi-sensor and pressure plates and auditory feedback (ARAIG ,2013). The auditory feedback system may be augmented by an immersive external DIY 10.2 (or 22.2) (dome) surround sound system or simulated 10.2 headphones, the minimum number of speakers for spatial sensing to be heightened (Holman, 2001). Matching ultra high definition resolution, 22.2 surround sound systems are being developed to expand on spatial sensory experiences (Hamasaki, Nishiguchi, Okumura, Nakayama, & Ando, 2008). Olfactory simulation can be generated utilizing a Digi Radio linked to a device cloud service iDigi, utilizing the XBee Internet gateway XIG, and connected to a scent dispenser (Digi, 2013). Pronounced and exaggerated air movement actuated by directional fans with controllers linked to the measurement of a risk component can be made receptive to a sensory response to that air flow differential or natural wind in a hexagonal or surround array of fans and controllers (WindEEE Research Institute, 2010).

Game software engines can be built using the free open source (FOS) Gambit library for game theoretic simulations (McKelvey, McLennan, and Turocy, 2010). An alternative Java library named NECTAR has been built in a similar fashion to that of

Gambit (utilizing homotopy and gradient descent methods for calculating Nash equilibria) and can be considered as another foundation for building the study game engine. In a more efficient modified version of the gradient descent method of Govindan and Wilson (2009), Blum, Shelton, and Koller (2006) have produced a software library for computing equilibria in general topologies of games and will be included in the library to be used in this study's game engine.

In an attempt to introduce gamification of the interface to the game risk holodeck, a gamification engine may be utilized such as BunchBall's Nitro gamification software engine (BunchBall, 2013). The gamification of game risk analytics manifests the active participation of DMs in the guise of game environments that have transformed ordinary web interfaces into interactive game simulation interfaces. Gamification, more generally, enacts an environment of game dynamics and mechanics into non-game contexts, scenarios and presentations (Marczewski,. 2012). Nonetheless, this gamification of real time risk dynamics should be done in the spirit of gamefulness – empowering DMs to feel, in a multi-sensorial manner – a general and broad bird's eye view beyond visualization, of the evolution of strategy spaces in social-cosms, noöspheres of interaction. McGonigal (2013) defines gamefulness as the ability of games to present flexibility of needs and wants and to manifest environments that produce such feelings of achieving those wants and needs. Ludic interfaces may be more appropriate for a more playful or non-evasive mechanism to interact with risk dynamics of an organization (Gaver, 2009). An example of a ludic interface for risk analytics is a "teetering on the brink of disaster" scenario, where high reward is potentiated alongside high loss.

Perturbations along these “all or nothing” stratagem are often neighbors to more conservative stratagem (risk aversiveness) given that iterative amounts of new information have been introduced in a just-in-time fashion. The playfulness of garnering new insights (information and intuition) in real time may lead to different risk profiles for the DM and hence more potential for co-opetive strategy play (competitive “everyone wins something” outcomes).

The prototype risk holodeck for this study will then consist of a triangular array of Kinect 2.0 motion sensors synchronized on an Xbox One for more precise tracking of user movements navigating on a Virtuix omidirectional treadmill and hand and body gesturing at HD (or Ultra-HD) resolution and ultra-wide angle visualization on a Oculus rift HMD. An array of precisely controlled surround dome fans will simulate wind and temperature generation with an iDigi Xbee-based scent generator controlling olfactory output.

In an extension of a this risk prototype holodeck, role playing social risk environments can be simulated utilizing a series of singular holodecks. Consider an array of connected risk holodecks sharing in real time all aspects of information and data dynamics of an inception game involving an entire coalition space (inception and inceptee teams). An inception/inceptee team, a subcoalition of either, or an individual agent may be represented in a player holodeck. Subsets of these teams may be automated subgroups of agents based on a diverse uncertainty model regime. Therefore, game role playing agents (subteams) may be active real agents in a holodeck or part of an automated subteam.

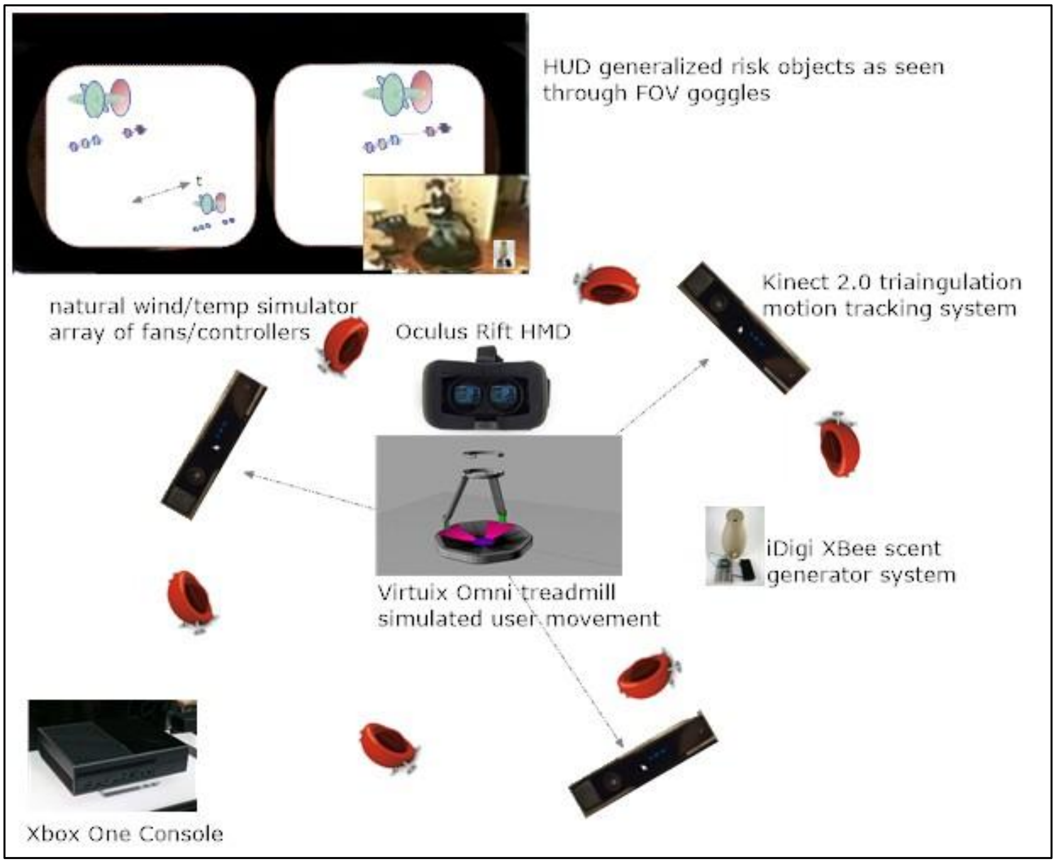

Figure 10 - Bi-lens view and layout of prototype DIY risk holodeck

Generators for other sensory system input are not currently economically practical and combinations of other senses are not well understood for their evolutionary purposes. Hence the initial concept prototype will be limited to these few sensorial outlays. Theoretically, as discussed earlier, a general game risk component can be mapped to a neural/nerve receptor so that a potential hyper-sensorium is possible. That each sensory system can be mapped to game risk components in a way that lends itself to a type of über-measurement or hyper-sensitivity of risk will be briefly outlined later in this section.

Inception games have been presented as metaphors for evolutionary social behavior in this study. Real time monitorization of such games can be visualized through the manipulation of general risk components connected to the game structure of inceptions given agent, payoff, and state evolution transition dynamics, and the physics of each inception level. Just-in-time analytics are able to then help predict patterns of future behavior, albeit with exponentially exploded limitations. Recently, in role-playing games, analytics has been applied to better understand the dynamics of development of that game. This quantitative approach to game evolution, gameplay, and game visualization has come to be labeled *game analytics* (Medler & Magerko, 2011). However, these analytics collect already played out dynamics. Little is produced in terms of pattern analysis and just-in-time prediction. The prototype presented here proposes that evolutionary patterns be gleamed from the running of inception game scenarios as what-if epochs, much as in the movie dynamics.

Inception levels are entered into a what-if scenarios where the parameters of risk and decision-making, such as payoff, coalition makeup, and state transition rules, along with the physical rules of engagement have been altered to emulate another possible future. In a reductionist sense, these what-if scenarios can be calculated in simple spreadsheet math. Multiple objective and agent games can also emulate such calculations in the operations research literature. This metaphor seems more fitting for multi-agent social games of engagement with dynamic agendas. Interaction with the model is as important as the calculus of risk for these social games. Hence, an immersion of involvement into the dynamics of inceptions is important to the psychological connection to decision-making in society.

While anthropomorphic or mammalian sense spectra will need to be appropriately matched to the spectra of game risk components, a related development is the augmentation and hyper-sensitization of those very same sense spectra – transhumanism (H+) and Kurzwelian singularities (Kurzweil, 2006; 2012; More, 2013). Khamis, Jones, Johnson, Preti, et al. (2010) have recently developed digital scent olfactory augmentation with a conceptual prototype of 2,000 olfactory chemical receptive sensors (canine range) compared to 400 human nasal sensors. In Yang, Coll, Kim, & Kim (2011) vision augmentation is proposed to hyper-track phenomena with multiply-connected cameras and information fusion, analyzing the understanding of complex scenery – a feat that is beyond the limits of current human visual processing with field of view, angular resolution, visible spectrum, and blind spot limitations. Sensory systems may be extended by proxy, (i.e., mapping extrasensory ranges of one sense to normal sense ranges of another). In this vein, Thomson, Carra, & Nicolelis (2013) have coined the term *sensory neuroprostheses* to mean the connection of existing hominid neuro-transmission sites for sense receptors to spectra of some phenomena, such as ultra-violet light that are not detectable by receptors in normal hominids with wavelength ranges between 390 and 750 nanometers. Recently, Hauner, Howard, Zelano, and Gottfried (2013) have shown that by introducing multiple sensory feedback via an added scent stimuli, fear extinction was possible in slow-wave sleeping individuals monitored by fMRI. The implications for a fuller sensorium of induced feedback flows points to the potential for sharper decision processing training without or at least with attenuated risk aversion when none is causally linked to objective results.

Future Considerations

Inception game models can be further generalized by expanding on more complex and realistic scenarios in real world conflict, government, politics, and economies. For example, the rules of inception levels can be expanded to include limiting cases of limbo, (i.e., the asymptotic behavior of equilibria and game solutions as $l \to \infty$ where l is the number of embedded inception levels) with the prospect of possible convergence of game solutions to $\varepsilon -$Nash equilibria or $\varepsilon -$evolutionary stability as well as to other forms of game solutions and equilibria discussed in the appendices. Computational time dilation schema may be made more fundamental for strategy development, (i.e., synchronized near-parallel time dilation in higher level inceptions may manifest the development of more advantageous game strategies through faster search or computation of game solutions).

The computation of *risk regions* of stability for games is a new approach that ties the topology of agent or collective coalition team risk spectra expressed in phase space (as in chaos theory) to game solution stability regions may be investigated. Risk regions are multi-dimensional risk profiles for agent DMs. In this maneuver, regions of risk spectra that agents display can lead to different manners in which game solutions may converge or diverge. In this way, game stratagem can be shaped or manipulated in order to provide stable game solutions that are advantageous to subsets of agents.

Inceptions are a very general approach to conflict games involving dynamic information transformation and transit between agents. Games that involve information imperfections in the form of transient noise (both deliberate or unavoidable), inter-agent and inter-coalition exchanges, or natural distortions (common knowledge among all agents that leads to noncausal biases) have been studied in isolation. By combining all forms of disinformation in inceptions, a more general model for all games sculptured in the mold of generalized statistical games can be made. For example, in Teng (2012), a linear Gaussian noise model is utilized to conceptualize an iterative Bayesian update schema (Bayesian equilibrium iterative conjectures – BEIC) for forming a convergence to prior distributions (and hence a-posteriori distributions) of strategies to build mixed stategy stable solutions (equilibria) to games with imperfect information that are expressed as general noisy games. This schema starts with an initial uninformative prior distribution in the form of a uniform distribution of strategies. By extending and generalizing the game noise model to nonparametric cases or even more general information geometric concepts of noise in probability spaces, inceptions may be made more expressive (Amari and Nagaoka, 2007). Even in emergent game theories as discussed here, apriori probability distributions are given for the development of stable solutions and equilibria.

The prospect of using information criteria(IC) that depend on a distance metric (or divergence) in the geometry of probability spaces for statistical estimates, such as the before mentioned divergence families, (i.e., KL and *f*-divergences) with suitable properties of parsimony and robust model fitting, applied to strategy estimation, may point to more robust estimates of distributions of strategies and hence of better equilibria and game solutions (Nakamura, Mees, and Small, 2006; Taniguchi and Hirukawa, 2012). In the case of noisy games, dynamic payoffs, and game state transition probability models such as generalized random processes, additional robust estimates for these game components may be computed based on IC-induced statistical functionals.

The development of graphical tree representations for belief revision-based dynamics for normal-form versions of inception games will be investigate. Simulations of a quantum-gravity interaction dynamic based on the causaloid framework in inception games can be done using general LQG or SuperString constructs. One may develop more efficient computational models for searching/calculating game solutions for inceptions, (i.e., Nash-like, evolutionarily stable, and alternative solution types introduced in this study). The development of graphical display models to simultaneously visualize/sense local (micro or super-micro) interactions with (super) macro and meso-level dynamics in inception games should be investigated as a means of building a dynamically focused-based network view of inceptions. Finally, in a follow up study, real-world scenario data using mixtures of classical and non-classical iuncertainty operators via GTU representations in inception games will be studied so as to compare obustness of the discussed inception models.

Conclusion

In this study, a novel approach to conceptually modeling a game theoretic structure involving deceit, coercion, and information extraction utilizing generalizations to mathematical social rules of engagement that involve multiple levels of inceptions (analogous to hypothetical lucid group dreaming), in the spirit of the psychological science fiction movie Inception, were constructed. Emergent game theories, including behavioral economics framed by quantum-gravity, general social, and generalized uncertainty games were used to show how spectral irrationality (rational/risk spectra), as manifested in psychologically mimicked recursive inception levels, can be modeled, with deception leading to attempts at information extraction from groups or individuals, resulting in possible versions of game equilibria and solutions. Inceptions were framed as generalized recursive social coalition games. Analogies were drawn between inceptions and emergent game structures with the hope of using powerful generalized uncertainty frameworks to express them.

Virtual world environments were posited as a conduit for performing simulations of these inception games as they can accommodate game strategic mixtures of automaton and human – two sources of interaction for the further study of inceptions for decision strategies in general social coalition games. The multi-dimensional nature of multi-sensorial interfaces between the human sensorium and risk manifolds in games, particularly in inception games, may be manifested in holodeck and holographic panoramas. This study presented possible DIY setups for such risk information immersions in experiencing decision games of the nature of inceptions.

Inception games are a conceptual abstraction for generalized physical-social interaction with resource exchange (inception and other utility tradeoffs). Diversity of uncertainty models and evolutionary dynamics can be included in inception game descriptions and representations. Simulations may be run based on mapping game dynamics to real-time risk sensorium for the decision-maker or coalitions involved in the risk theatre. Multi-dimensional and sensorial holodecks are ideal tools for mapping complex game dynamics to decision-making entities via risk connectives discussed in this study. Incpetion game generalities are representable by higher order meta-mathematics such as category/topos theory, automata theory, and biologics through risk sensorium mappings. Inceptions are novel ways to interpret general complex conflict scenarios independent of the domain of application, (i.e., military maneuvering, business ecocycles, government-social interaction, etc.). The lack of detailed simulation algorithm results in this study are to be accommodated in simulation/algorithm development, testing, collection of results, and comparative analysis in follow up studies.

Other conclusions based on inception-induced games are that strategies that emulate the rules of deceit and information extraction, as practiced in the concept movie Inception, are theoretical musings about what could happen in negotiation, diplomacy, and conflict in real world social co-opetive interaction. This meta-model for game conflicts does not attempt to formulate a general theory of social behavior between conflicting interests among governments, political groups, or competing coalition groups in industries. Many other factors are involved when emulating socio-economic dynamics. The hypothesis of this study's model may not generalize appropriately under more complex and restrictive action spaces, payoff dynamics, and engagement rules based on more formal negotiating methodologies. Inceptions are toy models for complex, adaptive, and emergent types of conflict games involving social agendas that are intertwined within multiple coalitions.

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Appendix A: Classical and Non-classical Decision-Game Theories

Decision Spaces

The classical approaches to decision behavior descend from the seminal expected utility (EU) theory of von Neumann & Morgenstern (1944). EU assumes rational behavior of the DM in the sense that they will pick the choice that will maximize their utility function. Worth of a decision is equated with the value of the utility in EU. The DM or agent is specified the following objects in the universe of discourse: (1) a set of available actions, $\mathcal{A}$, (2) a set of possible states of nature, $\mathcal{E}$, (3) an outcome associated to each action-event/state pair, $x\left(a_i, E_j\right), a \in \mathcal{A}, E \in \mathcal{E}$, (4) a function on *x*, *u* that measures outcome utility to a DM so that a preference order for that DM can be developed, (5) a measure of confidence or degree of knowledge with respect to the chance of occurrence of event/states, normally a probability or possibility measure, p, (6) a preference order for the DM, $\succeq$ for ordering the preference of alternative actions to be taken, (i.e., $f \succeq g$ means *f* is preferred over *g* or is at least as good as *g*), (7) a loss (-gain) value, *L* assigned to an outcome utility value, *u*, and (8) a criteria that assigns an action to be selected by an agent, $U : \mathcal{T} \times \mathcal{H} \to \mathcal{A}$ for (time, history) pairs, $(t, h) \in \mathcal{T} \times \mathcal{H}(t)$, where $\mathcal{H}(t)$ is the space of possible (history,action) path states from beginning recorded time, t_0 to the current time, t, $\left(E_{t_0}, a_{t_0}, E_{t_1}, a_{t_1} ..., E_t\right)$. Here, $U(t) = a_t$ is structured so as to satisfy some interest that the agent has with respect to its environment. In the case of classical EU-Bernoulli utility theory

$$U(t) = \arg\max_{a \in \mathcal{A}} \left[\int_e L_t\left(u\left(x(a,e)\right)\right) p_t(e)\, de \right] \tag{4.18}$$

the agent has self-interest to maximize its dynamic utility and the integrand is integrable (Lebesgue or Stieltjes-Riemann) so that (3.1) exists. In (3.1), past history is taken into account to reformulate the loss/utility function L_t and probability p_t at time *t*. This is the Bayesian interpretation of updating decision spaces. A posteriori probability distribution,

$$p_t(E \mid x) = \frac{p_t(x \mid E)\, p_t(E)}{\int_e p_t(x \mid e)\, p_t(e)} \tag{4.19}$$

following Bayes theorem with added information x and historically updated prior probabilities, refines $U(t)$ in the obvious fashion:

$$U(t \mid x) = \arg\max_{a \in \mathcal{A}} \left[\int_e L_t\left(u(a,e)\right) p_t(e \mid x)\, de \right] \tag{4.20}$$

In the simplest case, the utility function, *u*, is a real-valued function that preserves DM preference order, (i.e., $u(f) \geq u(g) \Leftrightarrow f \succeq g$). Nonetheless, for a given DM environment, the space of plausible utility functions, $\{u \in \mathcal{U}\}$ is not isomorphic to the space of preference orderings, $\{\succeq \in \mathcal{P}\}$. More directly, for some preference orderings, $\succeq_o$, no utility functions, u_0 may exist. This is no more ostensible than in modeling preference orders in cognitive and behavioral environments since in the classical EU-Bernoulli utility theory, event probabilities and outcomes are assumed to be known and preference orderings are well defined and transitive. Violations of these assumptions and adjustments made to EU-Bernoulli utility theory that claim to accommodate these scenarios will be reviewed and investigated later. In the case of a finite discrete state/event space, the traditional self-interested rational agent should choose to perform the action υ that maximizes the finite discrete expected utility:

$$EU(\upsilon) = \sum_i p_{E_i} U(\upsilon \mid E_i) \tag{4.21}$$

where E_i is an enumerated event, p_{E_i} is the objective probability of E_i happening, and $U(\upsilon \mid E_i)$ is the utility value of choosing action υ when E_i happens. Later Savage (1954) developed the subjective expected utility (SEU) in which subjective probabilities replace the objective ones in (3.4) and are tied to payoffs when real probabilities are unknown or have elements of uncertainty.

Simon (1957) disputed that all EU-based measurements of decision worth were valid because humans exhibit what he called bounded rationality – the limited ability of a human to make EU-type rational decisions based on inherent uncertainty or lack of information involved in making inferences pertaining to data, probabilities, events and outcomes, and risk measurement from data, time and psychological constraints. Moreover, inherent limitations in computational aspects of measuring utilities under constraints further defies rational decision processes. Instead, Simon proposed that the DM heuristically simplifies the utility function to achieve optimal values to reach, using more available information and calculations to satisfy their decision goal target. This process was famously labeled *satisficing*. Simon further enumerates a program for devising more accurate utility function spaces including, (i) vector valued multi-faceted utilities, (ii) accounting for information cost, and (iii) forming constrained utility function spaces that depict human-centric limitations and patterns of behavior. Simon's doctrine paved the way for further work in and the blending of ideas from behavioral economics and cognitively limited decision science.

Subsequent to the bounded rationality doctrine, Hixon (1987) showed that the near-optimal condition known as ε-rationality can better frame approximate real world suboptimal utilities. Essentially, if the optimal utility for a decision is given by u_{opt} then under certain conditions on a decision space, a family $\mathcal{U}_\varepsilon$ of ε-rational actions satisfying the condition $\left|u_{opt} - u(a)\right| < \varepsilon, \forall a \in \mathcal{U}_\varepsilon$ exists. The space of actions, $\mathcal{U}_\varepsilon$, under certain conditions of a corresponding multi-agent game, will generate corresponding ε-Nash equilibrium and ε-evolutionary stable strategies. Hence, if DMs are satisficed enough by taking actions in $\mathcal{U}_\varepsilon$ in certain co-opetive games, stratagem become stable, even under suboptimal bounded ε-rationality. This result had important ramifications for it implied that even utilizing tit-for-tat or under-cutting game strategies in an otherwise fair economic game, long-term co-opetive stability likely arose. In this paper we will denote such strategy or decision action types as ε-decision spaces.

Anscombe and Aumann (1963) developed an EU-based theory that attempted to accommodate both uncertainty and risk measurement of those actions with associated payoffs by connecting utility functions on both lotteries (probabilistic systems for distributing prizes) and rewards (risk amounts) to define more realistic subjective probabilities. These EU-based approaches to decision processing are based on rational decision making. As has been profoundly obvious, real-world decision processes are irrational on a spectrum. Concepts measuring ε-decision spaces in more varied and multi-dimensional ways, can form the basis for developing a decision-rational spectrum dependent on some definition of a vector rational parameterization, ρ (e.g. ρ may parameterize irrationalities stemming from risk, ambiguity, and uncertainty aversion metrics). We shall revisit this notion of decision spectra using a proposed VR visualization.

Kahneman and Tversky (1982:1984) and Kahneman (2011) and others have demonstrated that laden irrationalities take place in decision making under very general conditions of uncertainty, risk, and stress. Violations of the sure-thing principle (STP), Savage (1954), confound rational behavior in individuals who decide, in the face of a sure bet to gain something from a gamble, decide not to participate. The two-stage gambling problem and prisoner's dilemma are two major experimental confirmations of these violations. Additionally, violations of the independence axiom of EU theory where a probability factor applied to EU-equivalent options affects the choice made by individuals are known as the Allais paradox (Allais, 1953). Ellsberg paradoxes, the other major violation of rationality happens when the timing of presentation of a third action confounds the preference of a first action over a second action. Technically, if

$A, B,$ and G are actions such that $A \succ B$ (A is preferred over B), then it follows that $SEU(A') = SEU(A) + SEU(G) > SEU(B) + SEU(G) = SEU(B')$. In experiments this was shown to be G order dependent (G was presented to subjects first or not) to human DMs - the Ellsberg paradox (Camerer & Weber, 1992).

Clearly there are perceptual effects that transcend rationality assumptions for classical utility theories. Prospect theory as posed in Kahneman & Tversky (1984) attempts to take into account these ubiquitous paradoxes by assigning a realistic weight function, w to event probabilities, p_i and a value function, v to actions, as opposed to objective utility values, *u*. The overall prospect value for actions $\langle a \rangle = (a_1, a_2, ..., a_n)$ is given as:

$$U(\langle a \rangle) = \sum_i w(p_i) v(a_i) \tag{4.22}$$

The values for *w*, *v*, and *p* are edited in a pre-trial of outcomes before assignment in (3.5). In a sense, these edits are training sets akin to artificial neural network (ANN) learning. Notwithstanding the work to refine EU-based preferences for actions, subjective probabilities remain the anchor for such approaches and hence, the propensity to accumulate or perpetuate compound biases. The manners in which probabilities are assigned remain flawed or a more apt generalization to probability seems to be wanting in classical decision and game theories.

In Schmeidler (1989), measurement of vagueness is introduced and injected into the SEU methodology to more aptly handle human tendencies to linguistically smear the precise meaning of certain descriptive expressions when dealing with decision analysis. This approach develops the Choquet Expected Utility (CEU) in which global utility is defined as:

$$U(f) = A_S\left(u\left(f(s)\right)dv\right) \tag{4.23}$$

where A_s is a general aggregation operator, summing over the domain *s*. A_s is normally given as the Choquet integral over *s*. In this case, the Riemann-Stieljes integral is not consistent with non-additive measures. Here probabilities, *p*, are replaced by capacities (non-additive probabilities), *v*, that may not be additive, mirroring some human measures and descriptions of imprecise, incomplete, or approximate comparison. Capacities describe beliefs. Capacities, *v* satisfy the conditions:

1. $v(\varnothing) = 0$
2. $\forall a_i, a_j \in \mathcal{A}, a_i \subset a_j \Rightarrow v(a_i) \le v(a_j)$
3. $v(\mathcal{A}) = 1$

Capacities are also known as nonadditive probabilities and are correspondingly more general than classical Kolmogorov probabilities. Bayesian prior probabilities can be introduced to accommodate experiential information that may add to relevant information on decision making. Consider the following multiple priors generalization to CEU:

$$U(f) = A_C\left(\phi\left(A_s\left(u(f)dp\right)\right)dv\right) \tag{4.24}$$

where $p \in C$ is a possible prior probability from a space of priors, *C*, *v* is a probabilistic belief distribution over *C*, A_S and A_C are aggregate operators over the spaces, *S* and *C* respectively, and ϕ is a nonlinear function which measures the amount of aversion to ambiguity from the DM. Variational preference versions of (3.7) exist in which A_C takes on the role of a variational operator (minimization over a space of distributions over *S* with a Lagrange cost function) (Maccheroni, Marinacci, and Rustichini, 2005).

There is a multitude of manners in which to inject non-classicity into the domain of decision and game theories via the EU mechanism. One such non-classical approach is that of fuzziness which may be developed to model vagueness, imprecision, incompleteness of information, and linguistic preferences. Fuzziness can be introduced into decision spaces via the vagueness, imprecision, incomplete, or incomparability of probabilities, utility functions, and preference ordering or combinations thereof (Aliev, 2013). One may also consider the fuzzification (as well as the application of more general notions of uncertainty measures) of actions and outcomes as will be attempted here. We will return to describe these fuzzy decision models after a brief overview of generalized fuzzy measures that generate fuzzification of each of these components of decision spaces.

Non-additive probabilities (Choquet capacities), μ present with a generalization to classical probabilities when fuzziness is considered . μ may be sub(super)-additive depending on whether $\mu(\mathrm{g} \cup \mathrm{h}) \leq (\geq) \mu(\mathrm{g}) + \mu(\mathrm{h})$. Possibility measures are non-additive set functions such that $\mu\left(\bigcup_{i \in I} h_i\right) = \sup_{i \in I} \mu\left(h_i\right)$. Possibility measures model *inf* and *sup* operators for classical probabilities and complete ignorance or lack of information.

Through the *inf* and *sup* probability operators (generalizations to lower and upper bounds for probabilities, the *p*-boxes), possibility measures generate classical probabilities.

The dual of a possibility measure, μ is a necessity measure, η in which $\eta(h) = 1 - \mu(h^c)$, (i.e., if $\eta(h) = 1$, a necessary event, then $\mu(h^c) = 0$, its complement is an impossible event) . Fuzzy measures can be expressed as linear combinations of possibility and probability measures as in the class of so-called g_v measures which satisfy the additivity definition:

$$g_v\left(\bigcup_{i\in I} h\right) = (1-v) \vee_{i\in I} g_v(h_i) + v\sum_{i\in N} g_v(h_i), v \geq 0 \tag{4.25}$$

An even more general notion of fuzzy measure is given by the class of (*z*)-fuzzy measures that incorporate fuzzy number-valued fuzzy measures in order to more appropriately model linguistic expressions, imprecision, and imperfect information. Thus, a (z)-fuzzy measure, μ_z on $\mathcal{F}$ is a fuzzy number-valued fuzzy set function satisfying the following conditions:

1. $\mu_z(\varnothing) = 0$
2. $\mu_z(h) \leq \mu_z(g)$ if $h \subset g$
3. $\mu_z\left(\bigcup_{i\in I} h_i\right) = \lim_{i\to\infty} \mu_z(h_i)$ when $h_1 \subset h_2 \subset ... \subset h_n \subset ...,$ where $h_i \in \mathcal{F}$
4. if $h_1 \supset h_2 \supset ... \supset h_n,$ where $h_i \in \mathcal{F}$, $\exists n \ni \mu_z(h_n) \neq \infty_f$ then $\mu_z\left(\bigcap_{i\in I} h_i\right) = \lim_{i\to\infty} \mu_z(h_i)$

Here ∞_f is a fuzzy infinity where for any real M, $\ni \alpha_M \in (0,1]$ such that $\infty_f^\alpha < -M$ or $\infty_f^\alpha > M$. ∞_f^α is the α-level fuzzy set of ∞_f .

Fuzzy measures may then be utilized to fuzzify preference relations, utility functions, probabilities, actions, and outcomes. In this manner, fuzzy measures generate classes of general fuzzy decision spaces that span the mixture of these fuzzy components of a fuzzy decision space (Aliev, 2013). In order to pursue component fuzzy decision spaces, one must first define fuzzy actions, utility functions, preferences, and outcomes. We present definitions of these in the appendix for sake of continuity of discussion.

Choquet aggregation will be the foundation for fuzzy decision spaces. In Aliev (2013) behavioral decision spaces are expressed as combined states of fuzzy and possibilistic uncertainty states. We consider fuzzy state spaces of nature, $\mathcal{S} = \{s_i\}_{i=1,2,\dots N} \subset \mathcal{E}^n$ where $\mathcal{E}^n$ is the space of all fuzzy *n*-dimensional states of nature. Additionally, we model the fuzziness of states of the DM, $\mathcal{H} = \{h_i^f\} \subset \mathcal{E}^n$ and denote the space of fuzzy outcomes as a bounded set, $\xi \subset \mathcal{E}^n$. Form the combined state space, $\Omega = \mathcal{S} \times \mathcal{H}$. Next, we denote by $\mathcal{F}_\Omega$, the σ-algebra of subsets of Ω and consider the set of fuzzy actions, $\mathcal{A} = \{a \mid a : \Omega \to \xi\}$ as all $\mathcal{F}_\Omega$-measurable fuzzy functions from Ω to ξ. We now denote the decision space for DMs under imperfect information by $D_{f_i} = \left(\Omega, \xi, \mathcal{A}, \succsim_{f_i}\right)$ where $\succsim_{f_i}$ is the linguistic preference relation for a DM agent, *i*. This linguistic preference, $\succsim_{f_i}$ represents a general vagueness of comparison of worth of an action, *a*, taken under a fuzzy state of nature, $w=(s,h)$, with a certain outcome measured, $\zeta \in \xi$ that a boundedly rational DM agent *i*, operates under and that defines an order for the space of fuzzy utility function values. . The general fuzzy decision process is then the following: find the fuzzy action $a_f \in \mathcal{A}$ such that

$$U_f\left(a_f\right)=\sup_{a\in\mathcal{A}}\left\{A_{\xi_p}\left(u_f\left(a\left(s_f\right)\right)\right)\right\} \tag{4.26}$$

where A_{ξ_p} is an aggregation operator (such as the Choquet fuzzy integral) on the space of fuzzy actions, $\mathcal{A}$ with respect to the fuzzy measure ξ_p which is the equivalent to a linguistic (natural language-NP) probability distribution over fuzzy states s_f.

Rough set theory has also played a role in developing a framework for decision analysis and games with imperfect information. Pawlak (1982, 1994) developed the rough set theory in order to structure approximation spaces (lower and upper) to vague notions of attributes in information spaces. Later, probabilistic notions were introduced into rough sets in order to inject stochasticity into the parameterization of the approximation spaces of rough sets (Pawlik, Wong, & Ziarko, 1988). We briefly review rough set theory with respect to Bayesian approaches to decision theory from the proposals of Yao (2007). They represent a generalization to rough set theoretic notions for decision spaces of which we will categorize to include in a more generalized uncertainty approach to decision and game theories.

The probabilistic rough set theory begins with crisp algebraic rough sets. The rough set model is based on a quadruple, $(U, A_{tr}, \mathrm{V}, \mathrm{I})$ where *U* is a set of objects considered as the universe of discourse, A_{tr}, a set of attributes, $V=\{V_\alpha \mid \alpha \in A_{tr}\}$, a set of values of those attributes, and $I=\{I_\alpha \mid \alpha \in A_{tr}\}$, a set of information maps that assign a value to an attribute of an object, (i.e., $I_\alpha(x) \in V_\alpha$). The description of the object x, noted as $\operatorname{desc}(x)$, is given by a feature vector in the product space, $\underset{\alpha\in A_{tr}}{\otimes} V_\alpha$, (i.e., $\operatorname{desc}(x)=(v_1, \mathrm{v}_2 ...,)$). Now consider a pair, $(U, \simeq_e)$, called an approximate space in which U is as above and $\approx_e$ is a vague equivalence relation defined on U, (i.e., $\approx_e \subset U \times U$ can be considered as a indiscernable relation for element of U, a vague notion of comparing elements of U). Each $\approx_e$ defines a partition of U notated here by $U \mid \approx_e$. Consider an arbitrary element $x \in U$ and define the equivalence class subset (granule) of U containing x as $[x]_e=\{z \in U \mid x \approx_e z\}$. The set $[x]_e$ is then considered as the knowable description of x. Using our earlier definition, $\operatorname{desc}(x)=[x]_e$. $[x]_e$ defines the characteristics of any member of that equivalence set with respect to a feature vector profile from $\underset{\alpha\in A_{tr}}{\otimes} V_\alpha$. Rough set approximations of a set are then based on the space of these equivalence class descriptors in the following manner. For a subset $B \subset U$ and approximation space, $(U, \simeq_e)$, define its lower and upper approximation sets respectively as the following unary set-theoretic operators:

$$\begin{aligned} \underline{B}_{(U,\approx_e)} &= \left\{x \in U \,\middle|\, [x]_e \subseteq B\right\}, \\ \overline{B}_{(U,\approx_e)} &= \left\{x \in U \,\middle|\, [x]_e \cap B \neq 0\right\} \end{aligned} \tag{4.27}$$

Utilizing these set operators, any superset universe U may then be partitioned into three disjoint regions:

$$
\begin{aligned}
B^{POS}_{(U,\approx_e)} &= \underline{B}_{(U,\approx_e)}, \\
B^{BND}_{(U,\approx_e)} &= \overline{B}_{(U,\approx_e)} - \underline{B}_{(U,\approx_e)}, \\
B^{NEG}_{(U,\approx_e)} &= U \setminus \left\{ B^{POS}_{(U,\approx_e)} \bigcup B^{BND}_{(U,\approx_e)} \right\} = U \setminus \overline{B}_{(U,\approx_e)} = \left(\overline{B}_{(U,\approx_e)} \right)^c
\end{aligned} \tag{4.28}
$$

More generally, for a partition $\pi = \left\{ \pi_j \right\}_{j \in J}$ (index space J) of U, the regions can be expressed as:

$$
\begin{aligned}
\pi^{POS}_{(\alpha,\beta)} &= \bigcup_{j \in J} \pi^{POS}_{j(\alpha,\beta)}, \\
\pi^{BND}_{(\alpha,\beta)} &= \bigcup_{j \in J} \pi^{BND}_{j(\alpha,\beta)}, \\
\pi^{NEG}_{(\alpha,\beta)} &= U \setminus \left\{ \pi^{POS}_{(\alpha,\beta)} \bigcup \pi^{BND}_{(\alpha,\beta)} \right\} = \varnothing
\end{aligned} \tag{4.29}
$$

Depending on the logistics of the equivalence relation $\approx_e$, one may view these set operators as measures of the inside (elements definitely inside B), $B^{POS}_{(U,\approx_e)}$, outside (elements definitely outside of B), $B^{NEG}_{(U,\approx_e)}$ and those possibly either inside or outside, $B^{BND}_{(U,\approx_e)}$ of a set B. The last set, the boundary set, $B^{BND}_{(U,\approx_e)}$, is the source of uncertainty generated from the logistics of the equivalence relation, $\approx_e$. If $B^{BND}_{(U,\approx_e)} \neq \varnothing$, B is considered to be a *rough (imprecise) set*. Otherwise, B is considered to be a *crisp set*. Two types of rules for categorizing elements of $[x]_e$ into general partition sets can be developed based on the regions: (1) if $[x]_e \subset \pi^{POS}_{j(U,\approx_e)}$ then $[x]_e \xrightarrow{c=1} \pi_j$, and (2) if $[x]_e \subset \pi^{BND}_{j(U,\approx_e)}$ then $[x]_e \xrightarrow{0<c<1} \pi_j$, where

$c = \frac{\left|\pi_j \cap [x]_e\right|}{\left\|[x]_e\right|}$ is a *confidence* measure. The convergences, $\overset{c}{\to}$ are convergences in probability (confidence). Rule 1 depicts a deterministic one while rule 2 depicts a non-deterministic (probabilistic) one.

We next consider probabilistic rough sets using the Bayesian risk framework. One can start with a two state category problem (e.g., hypothesis testing) in which the states of nature are partitioned in two categories, $\Omega = \{B, B^c\}$. This situation may then be generalized to the finite (infinite index space *J*) partition problem,

$\Omega = \{\pi_1, \pi_2, ..., \pi_n\}\left(\left(\pi_j\right)_{j \in J}\right)$ using pairwise complementation (i.e.,

$\forall j \in J$, let $B = \pi_j$, and $B^c = U \setminus \pi_j = \bigcup\limits_{j \in J \setminus \{j\}} \pi_i$). Now consider the decision space components involved in a rough set setting. Let $\mathcal{A} = \{a_1, a_2, a_3\}$ be the space of actions where $a_{i,}$ *i=1,2,3* is the action in picking (classifying) one of the three regions, $B^{POS}_{(U,\approx_e)}, B^{BND}_{(U,\approx_e)}$, or $B^{NEG}_{(U,\approx_e)}$ respectively. Denote by $R\left(a_i \mid [x]_e\right)$ the risk (expected loss) associated with taking action a_i when $[x]_e$ is considered to be in a partition. A generalized risk can be expressed as:

$$R_A\left(a_i \,|\, [x]_e\right) = A_J\left(l\left(a_i \,|\, \pi_j\right) p\left(\pi_j \,|\, [x]_e\right)\right), \ \left(j \in J\right) \tag{4.30}$$

where $l_{ij} = l\left(a_i \mid \pi_j\right)$ is the loss (function) associated with taking action a_i when an element is in π_j, $p\left(\pi_j \mid [x]_e\right)$ is the probability that an element in $[x]_e$ is in π_j, and A_J is the generalized aggregation operator over the aggregation index space *J* and which defines the risk operator R_A. Bayesian minimum-risk decision rules can then be developed,

$$\begin{aligned}
&\text{if } R_A\left(a_1 \,|\, [x]_e\right) \le R_A\left(a_2 \,|\, [x]_e\right) \text{ and } R_A\left(a_1 \,|\, [x]_e\right) \le R_A\left(a_3 \,|\, [x]_e\right) \text{ pick } B^{POS}_{(U,\approx_e)}, \\
&\text{if } R_A\left(a_2 \,|\, [x]_e\right) \le R_A\left(a_1 \,|\, [x]_e\right) \text{ and } R_A\left(a_2 \,|\, [x]_e\right) \le R_A\left(a_3 \,|\, [x]_e\right) \text{ pick } B^{NEG}_{(U,\approx_e)}, \\
&\text{if } R_A\left(a_3 \,|\, [x]_e\right) \le R_A\left(a_1 \,|\, [x]_e\right) \text{ and } R_A\left(a_3 \,|\, [x]_e\right) \le R_A\left(a_2 \,|\, [x]_e\right) \text{ pick } B^{BND}_{(U,\approx_e)}
\end{aligned} \tag{4.31}$$

Probabilistic approximation sets and regions based on the partition sets can then be given as:

$$\begin{aligned}
\pi^{POS}_{j(\alpha,\beta)} &= \left\{x \in U \mid p\left(\pi_j \,|\, [x]_e\right) \ge \alpha\right\}, \\
\pi^{BND}_{j(\alpha,\beta)} &= \left\{x \in U \mid \beta < p\left(\pi_j \,|\, [x]_e\right) < \alpha\right\}, \\
\pi^{NEG}_{j(\alpha,\beta)} &= \left\{x \in U \mid p\left(\pi_j \,|\, [x]_e\right) \le \beta\right\}
\end{aligned} \tag{4.32}$$

$$\begin{aligned}
\underline{\pi}_{j(\alpha,\beta)=} &= \pi^{POS}_{j(\alpha,\beta)}, \\
\overline{\pi}_{j(\alpha,\beta)=} &= \pi^{POS}_{j(\alpha,\beta)} \cup \pi^{BND}_{j(\alpha,\beta)} = \left\{x \in U \mid p\left(\pi_j \,|\, [x]_e\right) > \beta\right\}
\end{aligned} \tag{4.33}$$

Finally, the three rough set regions based on a full partition, $\pi = \left\{\pi_j\right\}_{j \in J}$ can be expressed:

$$\begin{aligned}
\pi^{POS}_{(\alpha,\beta)} &= \bigcup_{j \in J} \pi^{POS}_{j(\alpha,\beta)}, \\
\pi^{BND}_{(\alpha,\beta)} &= \bigcup_{j \in J} \pi^{BND}_{j(\alpha,\beta)}, \\
\pi^{NEG}_{(\alpha,\beta)} &= U \setminus \left\{\pi^{POS}_{(\alpha,\beta)} \bigcup \pi^{BND}_{(\alpha,\beta)}\right\}
\end{aligned} \tag{4.34}$$

where $\alpha = \dfrac{l_{12} - l_{32}}{(l_{31} - l_{32}) - (l_{11} - l_{12})}, \beta = \dfrac{l_{32} - l_{22}}{(l_{21} - l_{22}) - (l_{31} - l_{32})}$ and W.L.O.G., it is assumed that (i) $l_{11} \le l_{31} < l_{21}$, (ii) $l_{22} \le l_{32} < l_{12}$, (iii) $(l_{12} - l_{32})(l_{21} - l_{31}) > (l_{31} - l_{11})(l_{32} - l_{22})$ and for simplicity that $l_{ij} = l_{ik}$, $\forall j \ne k$, and $i = 1, 2, 3$. The associated rules can then be expressed as: (1) if $[x]_e \subset \pi^{POS}_{j(U, \approx_e)}$ then $[x]_e \xrightarrow{c > \alpha} \pi_j$, and (2) if $[x]_e \subset \pi^{BND}_{j(U, \approx_e)}$ then $[x]_e \xrightarrow{\beta < c < \alpha} \pi_j$, where

$c = \dfrac{|\pi_j \cap [x]_e|}{\|[x]_e|}$ is a prior mentioned *confidence* measure.

Fuzziness is but one class in a larger family of uncertainty measures that transforms a classically structured problem, in our case, a decision (game) space to a more general one involving the 3-space spectra of decision-making, (risk, uncertainty, ambiguity). We will return to an approach to express and define this family. Before this another class of non-classical approaches to decision spaces will be reviewed and framed for our situation.

Recently, approaches from quantum probability have been applied to decision theories using both the Everittian many-worlds interpretation (MWI) from Wallace (2009) and the more traditional Schrödinger wave equation collapse from Busemeyer, Wang, & Townsend (2006) , Busemeyer, Wang, & Lambert-Mogiliansky (2009), and La Mura (2009). In the MWI approach, a quantum state Ψ is expressed as the superposition:

$$|\Psi\rangle = \sum_i c_i \,| r_i \qquad (4.35)$$

where $|r_i\rangle$ is the i^{th} decision branch outcome receiving a reward of r_i. The universe of probabilities still holds, (i.e., $\sum_i |c_i|^2 = 1$) and the quantum weight of the i^{th} branch (future self in the i^{th} branch world) is given by the amplitude squared, $w(r_i) = |c_i|^2$. This is also the Born rule for the occurrence of a multi-world branch in the MWI. For a game strategy, υ in a game, G,

$$\upsilon(G) = \sum_i w(r_i)\upsilon(r_i) \tag{4.36}$$

is the quantum utility with the appropriate quantum weights and rewards for the game environment. In a naïve rational setting, an agent prefers playing game *A* to game *B* if $\upsilon(A) > \upsilon(B)$. Denote by $\succeq_a^{\psi}$ a preference order for possible games for agent *a* when in quantum state ψ, (i.e., $A \succeq_a^{\psi} B$ means game *A* is preferred to game *B* by *a* when in quantum state ψ). The Born rule theorem then states that there exists a quantum utility, υ as in (3.19) such that $\succeq_a^{\psi}$ is uniquely defined (up to affine transformations on the space of games, Γ_a for agent *a*) by υ. This dictum is the centerpiece of a quantum rational decision theory in the face of the MWI and descends from the EU theory. Using certain consistency criteria, which include the concepts of diachronicity and solution continuity, to be described more technically below, Wallace (2009) showed that in the case of the MWI, the Born rule sets the precedent for a rational quantum behavior.

Diachronic consistency stipulates that if for MWI descendents, a_i of a seed agent, a, $A_i \succeq_{a_i}^{\psi} B_i$, where A_i precedes A and B_i precedes B for the seed agent, then for the seed agent a, $A \succeq_a^{\psi} B$. Define a divergence measure, $D_a : \Gamma_a \times \Gamma_a \to \mathbb{R}^+$ for the space of games, Γ_a available to play by an agent or society of agents, a. The solution continuity condition states that if $D(A_\varepsilon, A) < \delta$ and $D(B_\varepsilon, B) < \delta$ for sufficiently small $\delta > 0$, then $A_\varepsilon \succeq_a^{\psi} B_\varepsilon$ when $A \succeq_a^{\psi} B$. This can be generalized to continuous-valued decision branching problems assuming a normed topology in Γ_a. Nonetheless, there are problems with (i) the conditions under Wallace (2009) for which the Born rule applies and (ii) in light of behavioral economics and irrationally masked decision processing in humans, the violation of order preferences such as transitivity, rationality assumptions cannot be made.

The MWI utilizing the Born rule and interpretation is a deterministic approach to quantum mechanics. We now review more traditional quantum probability approaches to decision processes. In Pothos & Busemeyer (2009), a quantum decision model outperformed a more traditional Bayesian Markov model in predicting the decision behavior of individuals participating in two-stage gambling and prisoner's dilemma games – two of the major empirical tests of the existence of widespread violations of the STP. In La Mura (2009), quantum probability is used to account for a major variant of irrational decision behavior known as the Ellsberg paradox. In general, most information received by real world DM is of the incomplete or uncertain nature. These twin perturbations of rational theory dictate that another approach be taken. In order to form a more comprehensive framework for games, one must consider a more general theory for decision, relevant in nonclassical settings such as QM, relativity, quantum gravity information and irrationality.

Busemeyer and Bruza (2012) review various contemporary attempts made to overlay a quantum probabilistic decision framework for cognitive decision making to more adequately model irrationality while simultaneously generalizing the mathematical approach using Markov processes. Trueblood & Busemeyer (2012) propose a quantum probability model for causal reasoning that can be utilized to describe interference effects brought on by irrational decision making. These models are not intended to propose a quantum model for the hardware of the brain. They merely endeavor to more adequately predict or describe decision making under uncertainty when irrationality is displayed by DMs and within the current context and situation. Using results from earlier research on quantum cognition models from Pothos and Busemeyer (2009), Busemeyer, et al. (2011), Aerts (2009), Atmanspacher (2004) and Conte, et al. (2009), they further postulate that (i) before measurement , cognition can be described more by a wave than a particle manifested through human-induced ambiguity, (ii) current judgments affect the context of future judgments, and (iii) quantum logic by generalizing Boolean logic better models human judgments that do not obey Boolean constraints, and (iv) to make a current judgment, more than past history must be used – present contextual measurements must be made to resolve indeterminacy .

The technical crux of this argument is that in quantum reasoning, (i.e., complex human reasoning utilizing wave-interference and indeterminacy before measurement), the classical Bayesian probability theoretic requirement that

$p(A \cap B | H) = p(B \cap A | H)$ and hence that $p(H | A \cap B) = p(H | B \cap A)$, is violated. Quantum probabilities using a Hilbert space, replace classical sample spaces.

In is in this spirit that this research proposes that a spectrum of (ir)rationality for an agent or multi-agent society can be developed from patterns of order preference for games. These patterns then define equivalence classes of (ir)rationality within that spectrum.

Game Spaces

Games are extensions of decision spaces for multiple agents (one agent may be represented by nature or a state of a system as in decision theory). Classical games are categorized into three main types: (1) normal form or matrix games, (2) continuous static, and (3) differential. Matrix games depict a finite number of agents engaged in payoffs that are awarded after each round of decisions or actions that are taken from finite action spaces. Payoffs are represented in cells of a multi-dimensional matrix. Continuous static games have continuous payoff functions with static stratagem. Differential games are governed by differential equation systems (ordinary and partial) that describe the time-varying dynamics of payoffs and strategies (Vincent & Grantham, 1981). Multiple agents, indexed by I , engaged in games, each execute a stratagem space of actions $\{A_i\}_{i \in I}$, so that the strategy space of the game is given by the product space $A = \times_{i \in I} A_i$ in a rules-based manner such as sequentially or in an order depicting a ranking. After each round of actions are executed, payoffs $U = (u_i)_{i \in I}$ are calculated and distributed to each agent according to each agent payoff function, u_i. Payoff vectors are utility vectors computed for the agent space. Hence a matrix game can be depicted as the triplet, $G = (A, U, I)$, where I is a countable index space for counting game agents.

The interaction of multiple agents and their respective decision-making strategies (stratagem) vastly complicates the straightforward utility calculations in decision spaces. Agents endeavor to outguess each other's strategies and motives. This epistemic phenomena is known as an infinite regress and has been studied as an explicit epistemic logical problem subsuming game-theoretic equilibrium concepts (Hu & Kaneko, 2012). The more general infinite regress of *n* agents trying to recursively surmise the strategies of other agents we label in this study as *n*-guessing. Equilibrium as defined by Nash (1951), Hansanyi (1967/1968) endeavors to address the long-term stability of such *n*-guessing. Nonetheless, there are countless number of manners in which to form solution concepts or concepts of agent optimality in games (i.e., Pareto, minimax, maxmin, Nash, Nash ε-equilibrium, correlated equilibrium, and evolutionary stability).

Here we review and generalize where adequate, main elements of game theory. Let, $a_{-i} = (a_j)_{j \in I \setminus i}$ be the game profile strategy without agent i's strategy. Define mixed strategies (stratagem) for an agent i, $s_i \in S_i = \Pi(A_i)$ as members which are probability distributions of pure strategies which are the static actions in A_i played again. The stratagem space of a game, $S = \times_{i \in I} S_i$ is then the product space profile for the game agent population, I. The Bernoulli-EU utility for a game agent using stratagem $s = (s_i)_{i \in I}$ is defined as:

$$U_i(s) = \sum_{a \in A} U_i(a) \prod_{k \in I} s_k(a_k) \tag{4.37}$$

Def. A stratagem, *s Pareto dominates* strategy profile *g* (written $s\ p.d.\ g$) if

$\forall i \in I,\ U_i(s) \geq U_i(g)$ and $\exists\ \mathrm{k} \in I,$ such that $U_k(s) > U_k(g)$.

Def. A stratagem, *s* is *Pareto optimal* if there does not exist another stratagem $g \in S$ such that *g p.d. s*.

Pareto dominance (preferences) at best, defines a partial order for *S* since there may be multiple Pareto optimums. While Pareto dominance defines a partial ordering for game strategy profiles, it is not with respect to individual game agents. Best responses for an individual agent to the other agents' cumulative strategy define an individual's counter strategy to game subgroups,

Def. A *best response* of an agent *i* to the actions of others (a subgroup of agents $J \subseteq I$) is expressed as a general mixed strategy, $g_i \in S_i$, such that

$$U(\mathrm{g}_i, s(J,i)) \geq U(s_i, s(J,i)), \forall s_i \in S_i \text{ where } s(J,i) = (s_l)_{l \in J \setminus \{i\}}, J \subseteq I .$$

We denote this by g_i b.r. $s(J,i)$ for subsets of agents $J \subseteq I$. When $J = I$ one obtains the usual definition of best response.

Individual best response strategies are not very useful since in general, there exist an infinite number of them, except in the case of a unique pure strategy being the best response. Instead, the concept of Nash equilibrium is used,

Def. A strategy profile $s = (s_i)_{i \in I}$ is a *Nash equilibrium* for game $G = (A, U, \mathrm{I})$ if $\forall i \in I,\ s_i$ b.r. s_{-i}. If the *b.r.* is strict then *s* is a *strict Nash equilibrium*, otherwise it is a *weak Nash equilibrium*. If a unique Nash equilibrium strategy exists, it is called a *Nash solution*.

Nash equilibria are considered a measure of stability of a game since rational agents are not incentivized to change their long term strategies. Equilibria are shown using topological (Brouwer) fix point theory as a means to long-term stability (Nash, 1951). Weak Nash equilibria however are less stable since at least one agent can possibly become dominate with a strategy change. Hence weak Nash equilibria can accommodate possible rogue groups using other b.r. strategies aside from the Nash equilibrium. Nash (1951) showed that in a finite agent and action game there exists at least one Nash equilibrium. However, there is a number of stability equilibrium measures and solution concepts for games as previously mentioned based on the goal or preference agenda of the agent.

If an agent is concerned about maximizing their payoff when it may be surmised that other agents are playing their respective best response strategies in order to inflict the greatest damage or loss to that agent, then the *maxmin* strategy can be utilized, Def. The *maxmin strategy* of a agent *i* is given by: $\arg\max_{s_i}\min_{s_{-i}} U(s_i, s_{-i})$ and the maxmin value is $\max_{s_i}\min_{s_{-i}} U(s_i, s_{-i})$.

If an agent is concerned with minimizing another agent's maximum payoff without regard to theirs, the minmax strategy is employed,

Def. In an *n*-agent game, the *minmax strategy* for agent *i* against agent $j \neq i$ is the *ith* component in the mixed-strategy profile, s_{-j} in $\arg\min_{s_{-j}}\max_{s_j} U(s_j, s_{-j})$ with value $\min_{s_{-j}}\max_{s_j} U(s_j, s_{-j})$ for agent *j*. This strategy assumes that all agents in $\{-j\}$ are likeminded with respect to minimizing agent *j*'s maximum payoff.

For the simpler case of a finite, two-agent, zero-sum game; in a Nash equilibrium, the all agent payoffs coincide with their respective equal maxmin and minmax values. This is von Neumann's Minimax theorem. In a finite, two-coalition, zero-sum game, where the intra-coalition strategies are stable, a Nash equilibrium represents a stable payoff for each coalition equal to both its minmax and maxmin values. Inception games can be viewed (when stable) as two-coalition games and when the space of actions and recursions are finite, a Nash equilibrium signals the stability and equality of minmax and maxmin inception team payoffs. It is then crucial to understand when a true inception occurs and what the threshold for the consciousness-aware status of each coalition is as this is the proxy value for coalition payoffs in inceptions.

The maxmin strategy guides an agent towards optimal payoffs when their adversaries are knowledgeable about best responses. If instead, an agent wants to minimize their maximum regret in their decision (action) against another agent's actions, in the situation where no apriori knowledge of the other agent's knowledge of strategies is known (non-Bayesian) , then the concept of minimax regret is helpful.

Def. *Minimax regret* decision for an agent i where agents $\{-i\}$ take actions a_{-i} is defined by,

$$\arg\min_{a_i \in A_i}\left[\max_{a_{-i} \in A_{-i}}\left(\max_{a_i^* \in A_i} U\left(a_i^*, a_{-i}\right) - U\left(a_i, a_{-i}\right)\right)\right]$$

where A_i is the action space for agent i and A_{-i} is the action space for agents $\{-i\}$.

Note that mixed strategies in a profile s_i are linear (expectations) combinations of actions, a_i and so this definition can translate to mixed strategies without loss of generality.

As Pareto dominance defined a partial ordering of dominance among individual agent strategies, the concept(s) of (strict, weak, very weak) dominance define an order for agent strategies against the strategy profile of all other counterpart agents in a game.

Defs. Consider two strategies s_i and s_i^* of an agent *i*. Then (i) s_i *strictly dominates* s_i^* if $\forall s_{-i} \in S_{-i},\ U_i(s_i, s_{-i}) > U_i(s_i^*, s_{-i})$, (ii) s_i *weakly dominates* s_i^* if $\forall s_{-i} \in S_{-i}$, $U_i(s_i, s_{-i}) > U_i(s_i^*, s_{-i})$ and for at least one $s_{-i} \in S_{-i},\ U_i(s_i, s_{-i}) \geq U_i(s_i^*, s_{-i})$, and (iii) s_i *very weakly dominates* s_i^* if $\forall s_{-i} \in S_{-i},\ U_i(s_i, s_{-i}) \geq U_i(s_i^*, s_{-i})$. A strategy s_i is *strictly* (*weakly, very weakly*) *dominant* if it strictly (weakly, very weakly) dominates all other strategies. A strategy s_i is strictly (weakly, very weakly) dominant for agent *i* if it *strictly* (*weakly, very weakly*) *dominates* any other strategy $s_i^* \neq s_i$ for that agent. A strategy s_i is *strictly* (*weakly, very weakly*) *dominated* for an agent *i* if another strategy s_i^* strictly (weakly, very weakly) dominates s_i.

Nash equilibrium can be generalized based on choosing strategies that are correlated with another probability distribution or decision rule. The method of iterated removal endeavors to remove in stages, successive dominated strategies by assigning the probability of selecting such strategies to 0. This is a finite stage process as well. This process also preserves Nash equilibrium and hence is a practical methodology of arriving at an equilibrium. When equilibria can be found, these games are called *dominance solvable games*. Nonetheless, it has been shown in Camerer (2003) and summarized in Robinson (2004) that in real world social settings a limited iteration of removal of dominant strategies takes place instead and hence if a rational agent proceeds in their thinking with a full iterated removal process, they will inevitably pick a suboptimal strategy. The average number of (limited) iterations done by agents in games found from experimentation was 3.8. More precisely, real world strategists used types of limited thinking strategies that were categorized by L_i-type agents who process *i* iterations. L_1 agents assume that the opponent uses a naïve uniform distribution for all pure strategies and then execute a *b.r.* strategy. L_2 agents assume their opponent is an L_1 agent and so on. If we denote by L_k^{BE} that class of agents in L_k that act as behavioral economists using the strategies that yield the highest payoffs and those that use equilibrium strategies as L_k^E in L_k, then in simple games $\left|L_k^E\right| >> \left|L_k^{BE}\right|$, whereas in complex games $\left|L_k^{BE}\right| >> \left|L_k^E\right|$.

Def. In *n*-agent game $G = (A, U, I)$, a *correlated equilibrium* is a tuple (v, π, σ), where $v = (v_1, v_2, ..., v_n)$ is an *n*-tuple of random variables with associated domains $D = (D_1, D_2, ..., D_n)$, π is a joint prob. distribution over *v*, and $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ is a vector of mappings $\sigma_i : D_i \rightarrow A_i$, such that for each agent *i*, and all possible mappings $\sigma_i^* : D_i \rightarrow A_i$ we have that,

$$\sum_{d \in D} \pi(d) U_i(\sigma_1(d_1), ..., \sigma_n(d_n)) \geq \sum_{d \in D} \pi(d) U_i(\sigma_1^*(d_1), ..., \sigma_n^*(d_n)).$$

Again, this definition is generalizable to strategies from actions. Given a Nash equilibrium, a corresponding correlated equilibrium can be constructed. However, not every correlated equilibrium has a corresponding Nash equilibrium. It is anticipated that the findings of possible Nash equilibria in games structured as inception-like levels of deceit and information extraction will illuminate the idea of constructing better decision-theoretic tools for complex negotiation in situations involving uncertain alliances and coalition as exists in many geo-political and national legislative agendas. Nash equilibrium and the spectrum of Nash ε-equilibria in games translate into situations where all game agents are rationally persuaded to not change their respective stratagem after some decision stage point in the game evolution, (i.e., no rogue or wildly renegade irrational behavior occurs after a specific point in game time). We next define the concept of ε-Nash equilibrium in which agents are willing to instigate suboptimal strategies (worst than *b. r.*) if the gains from such equilibrium solutions are only minimally better and do not outweigh the resources needed to instigate a *b. r.*.

Def. Let $\varepsilon > 0$. The strategy profile $s = (s_i)_{i=1,\dots,n}$ is an ε-*Nash equilibrium* if for all agents *i*, and all strategies $s_i^* \neq s_i$, $U_i(s_i, s_{-i}) \geq U_i(s_i^*, s_{-i}) - \varepsilon$.

ε-Nash equilibria always exist and every Nash equilibrium is in a neighborhood (using the strategy profile space norm topology) containing ε-Nash equilibria. Nonetheless, the existence of ε-Nash equilibria does not imply the existence of a nearby Nash equilibrium. From a computational point of view, ε-Nash equilibria may represent many exactly computed Nash equilibria based on computational error. If $R_{s_i}(\varepsilon)$ depicts the resource differential between executing an ε-Nash equilibrium strategy s_i^{ε} for agent *i*, and a Nash equilibrium strategy s_i^N, for that agent, then if $U_i\left(s_i^{\varepsilon}, s_{-i}^{\varepsilon}\right) > U_i\left(s_i^N, s_{-i}^N\right) - \left(R_{s_i}(\varepsilon) + \varepsilon\right)$, it will be more advantageous to execute the ε-Nash equilibrium than the Nash equilibrium. The condition assumes that the resource values are somewhat statically calculated and are not taken into account in the calculation of the payoff value, (i.e., payoffs are not profits). In the case where errors in strategy execution are likely but small, the concept of perfect equilibrium is germane. When more than one (ε-) Nash equilibrium exists for a game, a *Schilling focal point* is a (ε-) Nash equilibrium that is advantageous to others in some sense, normally has uniformly larger payoffs for each agent (*payoff dominant*) or a smaller maximum loss for each agent (*risk dominant*). (ε-) Coordination games are those in which when two (or more) agents choose the same strategy, two or more (ε-) Nash equilibrium occur.

Def. A mixed strategy *s* is a *perfect equilibrium (trembling-hand)* of a normal-form game *G* if there exists a sequence $S = \{s_i\}_{i=1,2,\ldots}$ of mixed strategy profiles such that $\lim_{n\to\infty} s_n = s$ in the strategy profile space norm $\| \ \|_S$, and that for each $s_m \in S$ and agent *i*, s_i b.r. $(\mathrm{s}_m)_{-i}$.

Perfect equilibrium is a stronger condition that Nash equilibrium and indicates a kind of robustness against small errors in the judgment or execution of strategies of agents. Essentially, if one were to depict a small perturbation, $\delta > 0$ in strategy profiles, then given a perfect equilibrium *s*, there is a sequence of strategy profiles S of which a member $s_i \in S$ will be within that perturbation of *s*, $\|s - s_i\|_S < \delta$, and additionally, still be a best response to any other given strategy $s_m \in S$ in that sequence.

Evolutionarily stable strategies will be defined next. These strategies are resistant to new outside or unknown strategies being introduced to a game and dominating after a period of time.

Def. (coalition version) Given a symmetric two-coalition normal-form game given by $G = \{\{C_1, C_2\}, A, U\}$, and $\varepsilon > 0$, a mixed strategy *s* is an ε*-evolutionarily stable strategy* (*EES*)$\Leftrightarrow$ for all other strategies *s**,

$$U\left(s, (1-\varepsilon)s + \varepsilon s^*\right) > U\left(s^*, (1-\varepsilon)s + \varepsilon s^*\right)$$

We have the following result: if *s* is an *ESS* for *G*, then (s, s) is a Nash equilibrium of *G*.

If, on the other hand, (s, s) is a strict Nash equilibrium for *G*, then *s* is an *EES*.

Appendix B: Emergent Decision and Game Theories

Decision sciences have been based on the concept of distance measures so as to distinguish between sub-optimal and optimal strategies using risk and utility based functions defined on strategy spaces. Classical approaches included Game Theory's minimax and related principles such as minimax regret, Decision Theory's Bayes and Admissibility principles, along with entropy based measures, and hybrids of these. What we have learned from our biological partners and the Universe (Nature) is that while evolutional processes seem at first sub-optimal, through iterative development they progress in increasingly optimal ways. Evolutionary games are specializations of a more general system of games, differential games, in which feedback differential systems govern the flow of the game. Controls are treated as the strategy functions for the agents of the game. In addition to differential games, which deal with continuous spaces of strategies and hence, of making decisions on a continuum, stochastic games are based on probabilistic systems, but with discrete spaces of strategies and decisions. Hybrids of both stochastic and differential games are then approached in which differential systems with probabilistic decision strategies control the game process.

Quantum Mechanics (QM) introduce a new concept of decision reality in that the quantum state superposition of pure and mixed strategies and decisions can be treated as generalized strategies. QM utilizing more general divergences introduced in the breadth section, may explain phenomena more fully than non-dynamic paradigms. QM hence draws us into the possibility of utilizing quantum distances (divergences) in developing a scheme for quantum decision processes.

Humans are imperfect in part because they interpret phenomena in fuzzy (imprecise) ways and react with suboptimal strategies. Much research has been done on the seemingly distinct way in which chaotic processes can describe natural phenomena in the large. Fractals arise from a kind of chaos, producing self-similar structures that emulate the shape and growth of most natural scaffolding. If a new decision process could simultaneously incorporate the idioms of fuzzy reasoning, evolutionary processes, quantum information, and dynamic stochastic systems, including the iterative chaotic processes and self-similarity of fractals, would this better explain how we interact with nature and each other? Would it be a better predictor of such dances of life or will it reveal that indeed probability rules absolutely or at least dominantly in between the realms of chaotic processes? Development of novel techniques for the calculation of decision processes using the properties of these emergent systems and their respective paradigms of life, inorganic and organic will be attempted by melting aspects of the emergent behaviors of quantum, fuzzy, evolutional, dynamic, and iterative complex systems, which include as special cases, chaotic and fractal behavior. The various components of this generalized game will interact holistically, but will display a fractal holonic inner-structure, i.e., one in which a successive “peeling back” of detail of sub-components will exhibit self-similarity, self-reliance, and meso-level dependence. Holons have been described as generic objects of the universe which depict properties of a whole and of a part simultaneously (Koestler, 1990). Holonic structures or holarchies are therefore the anti-thesis of hierarchies. Holism refers to the functional interconnectedness of each holon at every level of interdependence. In addition, generalized divergences, as discussed in the breadth section, may be used instead of more classical metrics to measure risk and loss in more general decision spaces, i.e., the quantum realm of states. We label this as an "emergent decision landscape", diagrammatically depicted in Figure 2 below. The common thread of information and decision flow will continue to be a generalized abstract game structure.

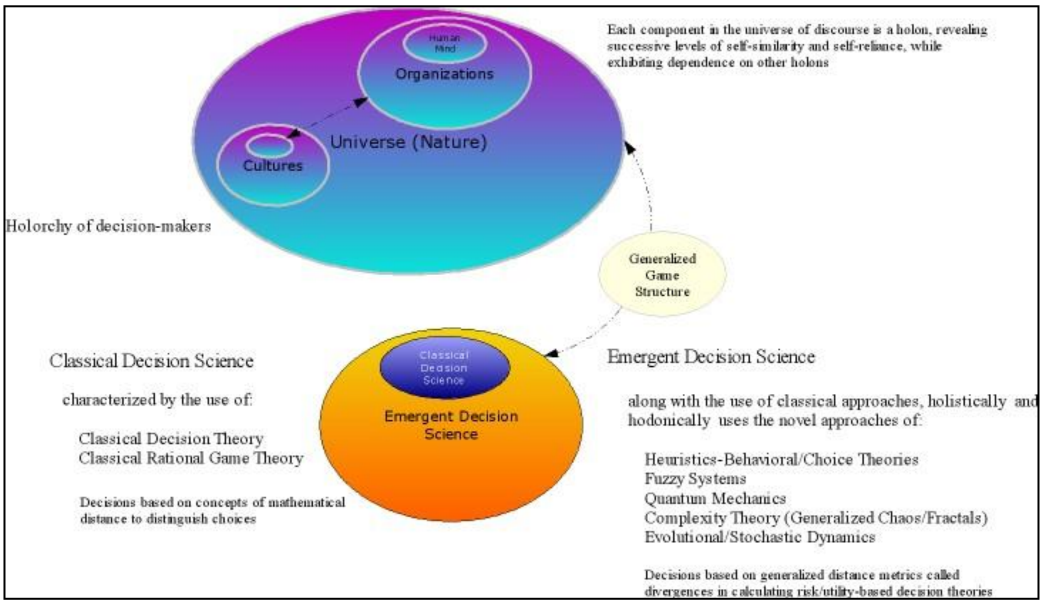

Figure 11 - Emergent decision landscape

Dynamic games will be reviewed first, which include differential, stochastic, stopping, and evolutionary flavors. Next, the emergent properties of fuzzy and quantum systems will be investigated in games. The case for a general chaotic game will also be approached. Hybrid games will then be proposed based on these systems. Human qualities of heuristics involved in aspects of behavioral game theory and neuroeconomics will be discussed briefly in anticipation of more detailed coverage and development in the application section.

Differential Games

When the time interval between stages of a multi-stage game decreases in the limit, as $\Delta t \to 0$, then agents must make decisions at each infinitesimal moment. If the decision space (strategy space) is continuous, the game changes continuously and the strategies can be defined as differential motion. These games evolve into differential systems.

Def. In a general differential *n*-person game between a sequence of *n* agents $z = \{z_i\}_{i=1}^{n}$, the dynamics are governed by a system of the form:

$$\dot{x} = \frac{dx}{dt} = f(t, x, z)$$
$$x(t_0) = x_0$$

with a payoff (cost) functional, $R_i(z) = g_i(t', x(t')) + \int_{t_0}^{t'} h_i(t, x, z)dt$, where $(t', x(t')) \in A$, intersects a set $A \subset R^2$ for the first time.

Agent *i* chooses a measurable control function $z_i(t)$ with values in some compact subset Z_i of the Euclidean space R^{p_i} such that $R_i(z)$ is minimized. The following conditions are also assumed in order to simplify the differential game for the existence of equilibrium strategies for (7):

(C_1) $f(t, x, z)$ is continuous in the set $[t_0, T_0] \times R^m \times (\otimes Z_i)$

(C_2) There exists a non-negative function $k(t)$ defined on $[t_0, T_0]$ with

$\int_{t_0}^{T_0} k(t)dt < \infty$, such that $|f(t,x,z)| \leq k(t)(1+|x|)$ for all $(t,z) \in [t_0,T_0] \times R^m \times (\otimes Z_i)$

(C3) For any $R > 0$, there exists a non-negative function $k_R(t)$ defined on $[t_0,T_0]$ with $\int_{t_0}^{T_0} k_R(t)dt < \infty$, such that $|f(t,x,z) - f(t,\bar{x},z)| \leq k_R(t)|x - \bar{x}|$ for all $(t,z) \in [t_0,T_0] \times R^m \times (\otimes Z_i)$

(C4) $g_i(t,x)$ is bounded in the set $[t_0,T_0] \times R^m$

(C5) $h_i(t,x,z)$ is continuous in the set $[t_0,T_0] \times R^m \times (\otimes Z_i)$ for each $i = 1,\dots,n$

Def. The unique solution $x^*(t)$ to (7), under conditions C1-C5 is called the *response* of the system to the agent controls $z(t) = \{z(t)_i\}_{i=1}^n$.

In order to define what an equilibrium strategy would be for (7), we first define a sequence of strategies on intervals of $[t_0,T_0]$ and develop sequences of corresponding games. This development will follow that from Friedman (2006).

Let *m* be any positive integer and let $\delta = \dfrac{(T_0 - t_0)}{m}$ define the half-open sub-interval partition width. Partition $[t_0,T_0]$ by the subintervals $I_j = (t_{j-1}, t_j]$, where $t_j = t_{j-1} + \delta$, $j = 1,\dots,m$. Denote the set of all control functions from the *k-th* agent that are defined on I_j by Z_j^k. Let $S_j^k(\delta)$ be maps that carry elements in $\underset{i=1,\dots,j-1}{\times}\ \underset{k=1,\dots,m}{\times} Z_i^k$ to elements in Z_j^k for $j = 2,\dots,m$. The vector $S^k(\delta) = ((S_1^k(\delta),\dots,S_n^k(\delta))$ is called a *lower* $\delta -$ *strategy* for agent *k*. The vector of vector strategies given by

$S(\delta)=((S^1(\delta),...,S^n(\delta))$ is called the *lower* δ *– strategy* for the differential game (7).

Now define a corresponding vector of control functions (strategies), $z=(z_1,...,z_n)$, with piecewise components z_k^j on the interval I_j as:

$$z_1^k = S_1^k(\delta),$$
$$\dots$$
$$z_j^k = S_j^k(\delta)(z_1^1,...,z_n^1,...,z_1^{j-1},...,z_n^{j-1})$$

z is referred to as the outcome of $S(\delta)$. The vector of cost or payoff functions can be written as $R(z)=(R_1(z),...,R_n(z))$. The game described above that depends on the interval width δ, is denoted by $G(\delta)$ and the sequence of games by $G=\{G(\delta)\}$. *G* is called the *differential game* associated with the differential system (7). $S(\delta)$ as defined above, is called a *strategy* for the game $G(\delta)$. Now suppose there exists a subsequence of $\{\delta\}$, labeled as $\{\delta'\}$, in which $\delta' \to 0$, as $m \to \infty$, and in which $S^k(\delta) \to s_k$ in the normed space $L^1([t_0,T_0])$, where s_k is some control function for agent *k*, and in addition, $\max_{t_0 \le t \le T_0} |x_\delta(t)-x(t)| \to 0$, where x is the trajectory corresponding to the controls $s=(s_1,...,s_n)$ and x_δ is that corresponding to *z*. Then *s* is an *outcome* of the strategy. Let $O[S(\delta)]$ denote the set of all outcomes of $S(\delta)$, the *outcome set* of $S(\delta)$. The set of all vectors $R(s)=(R_1(s),...,R_n(s))$ where $s \in O[S(\delta)]$ is called the *payoff (cost) set* of $S(\delta)$ and is denoted by $R[S(\delta)]$. Let $R_k[S(\delta)]$ denote the set of all *k*-components of vectors in $R[S(\delta)]$. We write this component-wise as $R_k[S(\delta)]=R_k[S^1(\delta),...,S^n(\delta)]$. Then the expression $R_k[S(\delta)] \geq M$ means that for any $\eta \in R_k[S(\delta)]$, $\eta \geq M$.

Def. The strategy, *S*, is a *Nash equilibrium strategy* of *G* if the set $R[S]$ is a singleton element in R^n, and if

$$R_k[S^1(\delta),..., S^{k-1}(\delta), S_*^k(\delta), S^{k+1}(\delta),..., S^n(\delta)] \geq R_k[S(\delta)], \text{ for } k = 1,..., n$$

for any other strategy $S_*(\delta) = (S_*^1(\delta),..., S_*^n(\delta))$ of *G*.

This defines an equilibrium strategy for the game *G* that corresponds to the system (7). In an entirely analogous way, one can define equilibrium strategies for different initial conditions in (7). Suppose we consider a different initial condition, $x(\tau) = \xi$, for (7). Let $\Omega = \{[t_0, T_0] \times R^m\} \backslash D$ for some exclusion set *D*. Define the following two spaces: $X(\Omega)$ is the set of points $(t, x(t))$ where $x(t)$ is any trajectory of (7) with the initial condition $x(\tau) = \xi$, and $(\tau, \xi) \in \Omega$, and $X^\varepsilon(\Omega) = \{x \in X(\Omega) :| x - 0 |< \varepsilon\} \cap [t_0, T_0]$. Let $u(t, x)$ be a function defined on $X^\varepsilon(\Omega)$ such that both *u* and $\nabla_x u$ are (1) bounded in $X^\varepsilon(\Omega)$, (2) continuous in *x* for each fixed *t*, and (3) measurable in *t* for each fix *x*. $u(t, x)$ is referred to as a *continuously differentiable pure strategy* for agent *k* in the game *G* corresponding to the system (7), utilizing the initial condition trajectory region $X(\Omega)$. Denote the corresponding strategy set by $S^k(\Omega) = \{S^k(\tau, \xi) : (\tau, \xi) \in \Omega\}$. Let $S(\tau, \xi) = (S^1(\tau, \xi),..., S^n(\tau, \xi))$. Now define the *Hamiltonian functions*:

$$H_i(t,x,z,p_i) = g(t,x,z)p_i + h_i(t,x,z)$$

where $p_i \in R^m$ and we write $p = (p_1,..., p_n)$. Let

$V(\tau,\xi) = (V_1(\tau,\xi),...,V_n(\tau,\xi)) = R[S(\tau,\xi)]$. In addition to the smoothness conditions placed on $u(t,x)$ above, let $u(t,x)$ also be *Lipschitz continuous* in (t,x). Now consider a new system of equations related to (7), the *Hamilton-Jacobi* equations of the differential game *G*,

$$\frac{\partial V_k}{\partial t} + H_k(t,x,u(t,x,\nabla_x V)) = 0, \text{ in } X^{\varepsilon}(\Omega), \text{ for } k = 1,..., n$$

subject to the condition $V_k = g_k$ on $\partial D \cap X^{\varepsilon}(\Omega)$.

Def. Let functions $u_i = u_i(t,x,p_i)$ for $i = 1,..., n$, where

$u(t,x,p) = (u_1(t,x,p_1),...,u_n(t,x,p_n))$ satisfy the following condition:

$$\min_{u_i^*} H_i(t,x,u_1(t,x,p_1),...,u_{i-1}(t,x,p_{i-1}),u_i^*,u_{i+1}(t,x,p_{i+1}),...,u_n(t,x,p_n)) = H_i(t,x,u(t,x,p))$$

Then u is called a *feedback control*.

Theorem. If in (9) above, *u* is a feedback control and are continuously differentiable in $X^{\varepsilon}(\Omega)$ and *V* is a twice continuously differentiable solution of (9), then $S(\tau,\xi) = (S^1(\tau,\xi),..., S^n(\tau,\xi))$ is a Nash equilibrium for any $(\tau,\xi) \in \Omega$ (Friedman, 2006, p.293).

Strategies corresponding to the solutions of the Hamilton-Jacobi equations of a differential game are then the Nash equilibrium strategy of that differential game. Numerical methods exist in finding solutions to the Hamilton-Jacobi equations under certain conditions, as outlined above and when such feedback controls exist. Feedback systems that tend to harness a system or game back into a realm of strategies or equilibrium that it began with are *negative equilibrating feedback* controls. Those that tend to change the regime of game equilibrium strategies from its initial configuration are called *positive equilibrating* controls or *deviation amplifying feedback*.

Stochastic Games

The first result on stochastic games was the work of Shapley in 1953 (Shapley, 1953b). The idea was to generalize a stochastic game to include a discounted factor, $\beta \in (0,1)$ in the description of the value of the game. Stochastic games are special versions of multi-stage games in which the dynamics of the game change from one move or stage to another. These games can occupy finitely many states, $s \in S$, where $|S| < \infty$ is a finite set of state indices. In more general cases, *S* may be denumerable or continuous and the number of agents may be countably infinite. Fink generalized the results of Shapley to *n*-agent games with countably many states, while Rieder generalized them to the case of countably many states (Fink, 1964; Rieder, 1979; Ummels, 2000). First, some notation and a definition of the general stochastic game:

Def. Let *X* and *Y* be Borel spaces. A *stochastic kernel* on *X* given *Y* is a function $P(.|.)$ such that: (a) $P(.|y)$ is a probability measure on X for each fixed $y \in Y$, and (b) $P(W|.)$

is a measurable function on Y for each fixed $W \in \mathcal{B}(X)$, the Borel σ-algebra of X. The set of all stochastic kernels on X given Y is denoted by $\mathcal{P}(X \mid Y)$. Let $\mathcal{P}(X)$ denote the set of all probability measures on X.

Def: A stochastic game is described as:

$$SG := \{\mathcal{N}, \mathcal{S}, (A_i, \Phi_i, R_i)_{i \in \mathrm{N}}, Q, T\}$$

where: (1) $\mathcal{N} = \{1, \ldots, n\}$ is the set of agents, (2) $\mathcal{S}$ is the state space, a Borel space, and (3) for each $i \in \mathcal{N}$, (a) A_i is a Borel subset of some Polish space (complete, separable, metric space) (and hence is also a Borel space) and is the set of actions for agent *i*, (b) $\Phi_i : \mathcal{S} \to A_i$ is a multi-function defined for each $s \in \mathcal{S}$ such that $\Phi_i(s)$ is the set of feasible actions for agent *i* at state *s*, (c) $R_{:i} : \mathcal{K} \to \mathcal{R}$ is a bounded measureable function that denotes the reward or negative loss of agent *i* given the state s and the action vector $a \in \Phi(s)$ and where $\mathcal{K} = \left\{(s, a); s \in \mathcal{S}, a \in \Phi(s) = \mathop{\times}_{i=1}^{n} \Phi_i(s)\right\}$, (d) $\beta_i \in [0,1)$ is agent *i*'s discount factor, (e) Q is a stochastic kernel in $P(\mathcal{S} \mid \mathcal{K})$ which specifies the game state transitions, and (f) $T \in \{0,1,2,\ldots,\} \cup \infty$ denotes the horizon or time length of the game (Dutta & Sundaram, 1997). Finally, let $M_i(s)$ denote the set of mixed actions (probability mixtures of actions) available to agent *i* when the game, *SG*, is in the state *s* and let $M(s) = \mathop{\times}_{i=1}^{n} M_i(s)$ be the composite space of mixed actions for all agents when *SG* is in state s.

At a particular state *s*, the agents of the game must play a regular game defined as $R(s) = r(a_1,...a_n;s)$. Transition probabilities, $p(s'|\, s, a_1,..., a_n) \in P(\mathcal{S} \,|\, \mathcal{K})$ are used to determine what state, s', the game will enter given that the agents use the actions $(a_1,...a_n)$ respectively on the next move, and the current state of the game is s. Generally, these transition probabilities will be determined by the complete past history of the game up until the current state. To this end, define the following:

Def. A *t-history* , h_t, of a game is a full description of the evolution (all previous states and action vectors) of a game up to time *t*, (i.e., $h_t = (s_0, a_0, ..., s_{t-1}, a_{t-1}, s_t)$). Note that h_t contains the current time state of the game, s_t. Let $H_t = \{h_t\}$ denote the set of all possible *t*-histories of the stochastic game *SG* for a given time *t*. For a given time *t* and *t*-history, h_t, $s[h_t]$ will be the state that results from the realization of h_t.

Markov processes and stationary strategies are often used instead to simplify the model. In a Markov stationary process, *p* is independent of time and any previous state and agent action vector. These are sometimes referred to as memory-less games. Let $R_i(t)$ denote the random variable (measureable function) representing the payoff to agent *i* from all other agents at stage *t* of the game. The expected value of $R_i(t)$, given the action vector $a = (a_1,...a_n)$ and current state *s*, $E_{(a,s)}[R_i(t)]$, is well defined for each agent *i*. We give the definition of a discounted payoff and stochastic game, Def. The β -discounted stochastic game, denoted by Γ_β , for $0<\beta\le 1$, has a payoff determined by (Connell, et al., 1999):

$$v_\beta^i\left(a,s\right)\left(t\right)\equiv\sum_{m=1}^{\infty}\beta^{m-1}E_{(a,s)}\left[R_i\left(t\right)\right] \tag{4.38}$$

Def. If $v_s^i\left(\beta\right)\left(t\right)=\sup_{a_i}\inf_{a_{-i}}v_\beta^i\left(a,s\right)\left(t\right)=\inf_{a_{-i}}\sup_{a_i}v_\beta^i\left(a,s\right)\left(t\right)$ for each agent *i*, then the *value* of Γ_β at time *t* is $v_s^i\left(\beta\right)\left(t\right)$. The optimal strategy vector (profile) for the *n* agents, $a^*=\left(a_1^*,...,a_n^*\right)$ at time *t* satisfies:

$$v_s^i\left(\beta\right)\left(t\right)=\inf_{a_{-i}}v_\beta^i\left(\left(a_1,...,a_{i-1},a_i^*,a_{i+1},,,a_n\right),s\right)\left(t\right)=\sup_{a_i}v_\beta^i\left(\left(a_1^*,...,a_{i-1}^*,a_i,a_{i+1}^*,,,a_n^*\right),s\right)\left(t\right) \tag{4.39}$$

for each agent *i*.

For the vector of states that a game progresses through, the vector $[v_s^i(\beta)]_{s\in S}$ is the value vector of Γ_β . Shapley proved that both $\left[v_s^i\left(\beta\right)\left(t\right)\right]_{s\in S}$ and a^* exist. Of interest is the behavior of Γ_β as $\beta\to 1$, (i.e., the asymptotically non-discounted stochastic game). In this limiting case, one can re-define the payoff determinant (4.38) as:

$$v_\beta^i\left(t\right)\equiv\lim_\tau\inf\left(\frac{1}{\tau}\right)\sum_{m=1}^{\infty}E_{(a,s)}\left[R_i\left(t\right)\right] \tag{4.40}$$

Mertens and Neyman (1981) proved that for this *limiting average* stochastic game, an optimal strategy and value $\left[v_\beta^i\left(t\right)\right]_{s\in S}$, a^*, both exist. More general stochastic games, such as *n*-person zero-sum β -discounted stochastic games with denumerable state spaces have been studied in which public signals (information vectors with relevant game information) given by a uniformly distributed sequence $\{\xi_n\}$, known to all agents, can be taken advantage of. These so-called correlated games can possess Nash equilibria and are said to have *correlated equilibria* under certain stochastic conditions (Nowak & Szajowski, 1999).

Time-discounted games are defined as discounted games where the discount factor is replaced by a function of time *t*. Hyperbolic time-discounted functions have become more popular than the initial use of exponential discounts because the former more accurately emulate the economic behavior of humans who reason that discounts increase less with at time periods that are closer to the present time when generally applied to time delays of rewards. These time-discounted functions take the form,

$$f_H(T) = \frac{1}{1+kT} \tag{4.41}$$

where *T* is the time delay from time 0 until payoff and *k* is a static damping factor. Read (2001) has shown in experimental tests that time-discounted factors are subadditive and do not increase with time, but rather are more dependent on the length of subdivided time intervals. In a general stochastic game with a subadditive time discount one has,

$$v_{k,r}^{i}(a,s)(t) \equiv \lim_{\Delta t \to 0} \sum_{m=1}^{\infty} \frac{E_{(a,s)}\left[R_i(t)\right]}{1+k(\Delta t)^r} \tag{4.42}$$

where $0 < r \leq 1$, *k* and is an empirical non-linear time perception factor and Δt is the time interval between payoffs. The corresponding limiting average case can be expressed as,

$$v_{\beta}^{i}\left(t\right)\equiv\lim_{\Delta t\to 0}\liminf_{\tau}\left(\frac{1}{\tau}\right)\sum_{m=1}^{\infty}\frac{E_{(a,s)}\left[R_i\left(t\right)\right]}{1+k\left(\Delta t\right)^{s}} \tag{4.43}$$

with optimal strategy $\left[v_{\beta}^{i}\left(t\right)\right]_{s\in S}$ and value a^*.

Stopping Games

Stopping games are another breed of stochastic game in which the strategy rules are based on optimal stopping times for stochastic processes. This discussion will center on the original proposal from Dynkin (1969). Here we let $\{X_n\}_{n=1}^{n}$ be a finite stochastic sequence defined on a fixed probability space (Ω, F, P). Let $F_n = \sigma(X_0, X_1, ..., X_n)$ be the σ-field generated by $\{X_n\}_{n=1}^{n}$. Define the random variable $\lambda(\omega)$ to be a Markov time with respect to the family $F = \{F_n\}, n \in N = \bar{N} \cup \{\infty\}$. λ takes on values in $\bar{N}$ and $\{\omega : \lambda(\omega) = n\} \in F_n$ for each $n \in \bar{N}$. $\lambda(\omega)$ is then a *stopping time* (finite Markov time) if $P(\{\lambda(\omega) < \infty\}) = 1$, i.e., if a stopping time exists in probability. Strategies in stopping games are the stopping times. Here $R_l(a) = X_{\overset{n}{\underset{i=1}{\wedge}} a_i}$ where $\overset{n}{\underset{i=1}{\wedge}} a_i = \min_{i=1,\ldots,n} a_i$, defines the payoff for agent *l* from all other agents in the game. Under certain regularity conditions, the value and optimal strategies exist for stopping games. In particular, under a payoff function of the form:

$$R_l(a) = X_{a_l} I_{\{a_l \le \min a_i\}} + W_{a_l} I_{\{a_l = a_i\}} + Y_n I_{\{a_l \ge \max a_i\}}$$

where $\{Y_n\}_{n=1}^{n}$ is another stochastic sequence measurable with respect to some increasing sequence in F and such that $X_n \le Y_n$ for each n, the existence of a game value and optimal strategies has been shown (Yasuda, 1985).

Evolutionary Games

Darwinian dynamics, the term used by evolutionary mathematicians to describe the processes that underlie natural selection, is a system of differential equations that endeavor to satisfy Darwin's conditions for evolution, including those of variability, heritability, reproduction, and survival. This evolutional process can be framed into a game, the evolutionary game. Unlike classical game theory where the emphasis is on the agent as a dynamical entity, the evolutionary game puts more importance in its framework on the evolution of strategies of species (agent groups). Agent groups come and go, but strategies of those groups dynamically change, based on the most simple dictums of evolution: (1) like tends to beget like, with heritable variation in between generations, (2) in a species of organisms, the prime operator is survival, and (3) heritable traits or the phenotype of the species influences the survival techniques imposed (Vincent & Brown, 2005). In an evolutionary game, two dynamics are in play simultaneously, population and strategies. Both these spaces can be described by a fitness function, generated by a kernel or a generating function. This is the most general form for a fitness function in that both population growth (decline) and strategy spaces can be realized by two separate instances of a single generating or G-function, as it has been labeled (Vincent & Brown, 2005). Game strategies in an evolutionary game are the species' phenotype, the characteristics trait of that group of organisms. Hence, within the same population group, the evolutionary trait is to have one G-function describe the strategies for each member of that group. G-functions will then describe the game strategies for a single pack of organisms. The same may be said for the population dynamic of that group.

Consider now the situation where groups of agents participating in a game are linked by common characteristics. In evolution, these may be a genotype, phenotype, or social dynamics, such as culture, religion, or nationalism. Blind evolution does not take into account adaptive learning. Learned evolution should incorporate an adaptive component. For our purposes, we review an evolutionary model that has stochastic components and a differential component that mimic the evolutionary trait. Strategy spaces for differential and evolutionary games are continuous. Species *j* adapts a stratagem that is denoted by the random variable $x_j(t)$ at time *t*. Let $\pi(x_j, t) = E_{\theta_j}(R(x_j, t))$ denote the expected payoff with respect to $G_j = G_j(\theta)$, the parametric distribution of strategies for that species. The evolutional rule is that a species will gravitate to strategies that will tend to produce larger values of π. In this regard, the tendency will be to pick a strategy that is in the direction of higher derivatives of π over time (Goercee & Holt, 1999). Since a certain amount of genetic or reproductive noise is present in every evolutionary process, a stochastic term must be added to this dynamic. Let f_j and $f_j^{'}$ be the population density and its time derivative respectively of species *j*. Consider the diffusion process described by the stochastic differential equation (SDE):

$$\frac{\partial G_j(x,t)}{\partial t} = -\pi(x,t) f_j(x,t) + \mu f_j^{'}(x,t)$$

This SDE is the Fokker-Planck equation from statistical physics. It has been derived for a very general noisy evolutionary process (Anderson, Goeree, & Holt, 1999). Other evolutionary systems involve dynamicism in the population and strategy spaces. Differential systems are given that express rate of change with respect to population density and strategy.

Def. A general dynamical system for an evolutionary game can be written in the form:

$$\dot{x}_i = x_i H_i(s,x,y)$$
$$\dot{y}_i = y_i N_i(s,x,y)$$

where $x = (x_1,..., x_{n_j})$, $y = (y_1,..., y_{n_j})$, and $s = (s_1,..., s_{n_j})$ are the population densities utilizing a strategy, the resources used, and the actual strategies respectively for the species *j*. Additionally, among the population of species *j*, there are x_i individuals who will utilize strategy s_i. $H_i(s,x,y)$ is a functional that describes the dynamics of the fitness of the evolutionary process. *H* replaces the concept of utility from classical games. $N_i(s,x,y)$ describes the dynamics of resource utilization (Vincent & Brown, 2005, p.97). Each s_i in itself may be a vector of strategies available to the x_i individuals utilizing them within the species *j* population, i.e., $s_i = (s_{i1},..., s_{in_i})$. In this case, one can derive the mean strategy for sub-population x_i. The different strategies employed within the same group of individuals contained in the species can be viewed as a phenotype. Note that time is not considered a differential in (5). The game defined by this dynamic system is given by Γ. To find a non-zero equilibrium solution for (5), involves either solving the system of equations

$$H_i(s, x, y) = 0 \text{, for } i = 1, \ldots, n_j$$

or choosing an initial condition $x(0)$ and iterate the corresponding difference equations until a convergence to a solution is reached. In order to further generalize the form (5) for a dynamic evolutionary process or game, we define a vector of fitness generating functions, $G = (G_i)_{i=1}^{n_j}$, for each species *j*, as follows:

$$G_i(v, s, x, y)|_{v=s_j} = H_i(s, x, y) \text{, for } i = 1, \ldots, n_j$$

where v acts as a variable holder for the species. This is known as a *G*-function and is isomorphic to the defining fitness function H via the relation (6). *v* is also known as a *virtual* strategy. Predator-prey and other more generalized competitive models of co-evolution can be described by this form. Let $N = \sum_{i=1}^{n} x_i$ denote the total population size of all species in the evolutionary game and define $p_i = \frac{x_i}{N}$ to be the strategy frequency for sub-population x_i. We can then use the vector $p = (p_1, \ldots, p_n)$ and *N* to re-define G_i, as $G_i(v, s, p, N)$.

<u>Definition</u> Evolutionary games are *density dependent* if:

$$\frac{\partial G_i(v,s,p,N)}{\partial N} \neq 0$$

or *frequency dependent* if:

$$\frac{\partial G_i(v,s,p,N)}{\partial p_i} \neq 0, \text{ for } i = 1,..., n_j$$

Evolutionary games may possess a special type of equilibrium in which a genotype depicted by a mixed strategy based on the probabilistic mixture of $s_i = (s_{i1},..., s_{in_i})$ is stabilized in the sense that another genotype mutant s_j is not able to evolutionarily invade and assimilate it out. The fitness generating function G determines the expected fitness of a species when the given virtual strategy *v* is employed by that species (Vincent & Brown, 2005, p.79). The differential system (5) may now by re-written in terms of the fitness generating function as:

$$\dot{x}_i = x_i G(v,u,x,y)\,|_{v=u_i}$$

$$y_i = N(u,x,y)$$

Each species may have a vector of strategies *u* and resources *y*, with vector resource function $N = [N_1, N_2,..., N_{n_y}]$, where n_y depicts the number of resources available to the species populations. Notation for all the above expressions will be in vector notation for *u*, *p*, *y*, and *N*. The more general setting is when an environment contains multiple species with multiple fitness landscapes. In this case there will exist multiple *G* fitness generating functions indexed by *i*. For us to express this multiple species buildup, we use the following notation: let r_i be the number of species given by adding each successive species population together. This can be expressed as the sum:

$$r_i = \sum_{j=1}^{i} n_{s_j} = n_s \text{ for } i = 1,...,n_g$$

where n_s is the total number of species in the environment.

Def. If $x^* = (x_1^*,..., x_n^*)$ is an equilibrium solution to an evolutionary system, and $s^*(s_1^*,..., s_n^*)$ is the corresponding mixed strategy vector then s^* is an *evolutionarily stable strategy* (ESS) for Γ if for any strategy, $s \neq s^*$ (corresponding to a mutant competitor),

$$G_i(s^*, x^*, y) = H_0 + (1-p)\Delta G(s^*, x^*, y) + p\Delta G((s_i^*, s_{-i}), (x_i^*, x_{-i}), y)$$
$$G_i(s, x, y) = H_0 + (1-p)\Delta G((s_i^*, s_{-i}), x\ , y) + p\Delta G_i(s, x, y)$$

then either

$$\Delta G_i(s^*, x^*, y) > \Delta G_i((s_i^*, s_{-i}), x, y)$$

or

$$\Delta G_i(s^*, x^*, y) = \Delta G_i((s_i^*, s_{-i}), x, y) \text{ and } \Delta G_i((s_i^*, s_{-i}), (x_i^*, x_{-i}), y) > \Delta G_i(s, x, y)$$

An evolutionarily stable strategy s^*, then either satisfies (1) the population x_i playing s_i^* does better playing against the coalition x_{-i} playing s_{-i}^* than any other mutant in the coalition does playing against x_{-i}^*, playing s_{-i}^* or (2) some mutant does just as well playing against *x* using s^*, but x_i using s_i^* does better playing against the mutant coalition x_{-i} using s_{-i}^* than any mutant in the coalition does against that mutant coalition.

General Dynamic Games

Deterministic, evolutionary, and stochastic differential games have been investigated in the prior sections. In this portion of the depth component we review general systems of dynamic games or games that involve a mechanism that is a general dynamical system. For stochastic dynamic games, the system may incorporate stochasticity in various ways, as in the involvement of a (Itô) stochastic PDE as the game control equation.

A dynamic system is a pair (M,Φ_t) such that M is a manifold of phase states and Φ_t is a time evolution operator, $\Phi_t : M \to M$, and that depends on time $t \in T$, for some time index space, and is smooth, i.e., $\frac{\partial^k \Phi_t}{\partial t^k}$ exists and is continuous for a sufficient number of $k \in \mathbb{N}^+$. One can view Φ_t as modeling or describing the evolution of a state space and hence an alternate definition of a dynamic system is a triple (S,T,R) where S is a space of states, T is a time index space, and R is a rule for mapping S into itself at different times. When $T = \mathbb{R}, \mathbb{R}^+, \mathbb{N},$ and $\mathbb{N}^+$, the system is called a *flow, semi-flow, cascade map,* or *semi-cascade map* respectively. Normally, in applications, Φ_t is the solution to a system of equations, such as: $H(x,\dot{x}) = 0$, where *x(t)* is a trajectory in M and $\dot{x}$ is the usually time differential of *x*. An initial condition will also be given and can be inherent in the equation $H(x,\dot{x}) = 0$. A general dynamic game can then be defined as a (stochastic) dynamic system in which a constraint condition $C[P(x)]$, is imposed on a payoff functional, *P(x)*. As in the description of a differential game, the game's response to the system is represented by one component of the trajectory *x(t)*. Other control mechanisms (for each agent and other possible non-agents) can be represented by other components of the trajectory. The dynamics of the game system are then governed entirely by the quad-tuple (M, H, P, C).

In the stochastic case, H will involve probability measures, as well as stochastic processes, such as Markov or Brownian processes. Generalized stochastic differentials may involve Itô partial derivatives or another defined stochastic derivative. In the deterministic case, if $H(x,\dot{x}) = \dot{x} - v(x)$ and $v(x) = Ax + b$, i.e., is linear, then $M \approx R^n$, i.e., *M* is isomorphic to an *n*-dimensional Euclidean space. *v* is called the *vector flow field*. If $A = 0_n$ (*n* x *n* zero matrix) and $b \neq 0$ then the trivial solution for the evolution operator is $\Phi_t(x) = x + bt$. When $b = 0$ and $A \neq 0_n$ then $\Phi_t(x_0) = x_0 e^{tA}$ at the initial point x_0 and, $\Phi_t(x)$ is generally determined by the eigenspectrum of *A*. In this case, the eigenbasis and eigenspectrum determine the stability of the solutions, i.e., whether they converge or diverge away from an equilibrium point at the origin. For two different initial conditions, x_0^1 and x_0^2, $d\left[\Phi_t(x_0^1), \Phi_t(x_0^2)\right]$ may converge or diverge (exponentially sometimes) depending on the eigenstructure of *A*. For a general dynamic system, *(H,M)*, the possibilities are infinite. There are "typical" systems that have been generalized, the so-called *structurally stable* (Morse-Smale) dynamic systems that lead to typical behaviors. Here "typical" will pictorially mean that in a phase portrait (the map of $\Phi_t(x(0))$ in the $(x, \dot{x})$-space as *t* changes), the trajectory will have one of four patterns, (1) source, (2) sink, (3) limit cycle (stable and unstable), or (4) saddle. Quasi-periodic behavior is a combination of periodic behaviors with differing frequencies, as in internal movement in a torus. It usually acts as an intermediate behavior between two typical behaviors in a trajectory's life. In a generalization of a result from Poincare and Bendixson, Smale showed that structurally stable dynamic systems lead to combinations of the typical behaviors (Smale, 1963). Structural stable is a global stability measure that essentially means that there exist a mapping that preserves orientation, differentiability, and topological invariance between two orbit trajectories (orbits) of two different evolution operators defining two nearby (both in a neighborhood of *M*) dynamic systems respectively. In a general dynamic game, the payoff constraint, $C[P(x)]$, may then define a general region (closed, bounded, convex, etc) in *M*. The range of the evolution, $R(\Phi_t)$, must then satisfy this constraint system *C*, i.e., the statement $C\left[P\left(\Phi_t(x)\right)\right]$ must be true.

Chaos is a particular type of dynamic systems behavior. Usually the term chaotic has meaning in terms of unpredictability of the long-term behavior of a dynamic system. Chaotic behavior can be exhibited in deterministic or stochastic systems that are linear or non-linear. It may also display itself in very simple iterative mappings such as the logistic function as the defining rule in a system. Therein has laid the recent popularity of chaos. More precisely, a dynamical system (M, Φ_t) is *chaotic* (dissipative dynamic) if:

(1) (M, Φ_t) is sensitive to initial conditions, i.e., $d\left[\Phi_t(x_0^1), \Phi_t(x_0^2)\right] \underset{t}{\rightarrow} \infty$ for two different initial points, x_0^1 and x_0^2 where $d\left(x_0^1, x_0^2\right)$ is arbitrarily small at some instant,

(2) (M, Φ_t) is topological mixing, i.e., $\exists$ a non-negative integer *N*, such that $\forall n > N$, and all pairs of open subsets in *M*, $A, B \subseteq M$, $\Phi_t(A) \cap B \neq \emptyset$, and

(3) the set of periodic trajectories (orbits) of (M, Φ_t) is dense in $C^{\infty}(M)$, i.e., any periodic orbit in (M, Φ_t) can be arbitrarily approximated well by a different periodic orbit of (M, Φ_t).

Attractors of *M* are subsets or regions in the phase space where trajectories that are sufficiently close to them are drawn into or converge into eventually in time. More precisely, *A* is an *attractor* of (M, Φ_t) if:

(1) *A* is invariant with respect to Φ_t, i.e., if $x^s \in A$ at some instant t, then $\Phi_t(x^s) \in A \ \forall t \in T$,

(2) $\exists$ a non-empty neighborhood of *A*, $B(A)$, the basin of attraction for *A*, defined as

$$B(A) = \bigcap_{\substack{N \text{ neighborhood} \\ \text{of } A}} \left\{ x_s \in \text{phase space of (M,}\Phi\text{)} : \exists t_N \in T \ \ni \ \Phi_t(x) \in A \ \forall t \geq t_N \right\},$$

and

(3) no subset of *A* satisfies (1) and (2), i.e., *B(A)* is the smallest neighborhood of *A* satisfying conditions (1) and (2).

A "sufficiently hard to describe" attractor set *A* is called a *strange attractor*. Two simple attractors are the fixed point (sink) and limit cycle sets. However, there may be attractors that are wildly formed and hence the name strange attractor. One of these is a fractal set. Fractals are sets which exhibit self-similarity and fractional Hausdorff dimension, a generalization to physical space dimension. Self-similarity is the property where as one "drills down" with more detail, the shape of the boundaries of the set are similar or identical to the shape of the higher , a view of that boundary. Popular examples of natural fractal sets are coastlines, ferns, tree limbs, and cloud structures. Formally, an attractor *A* is strange if (1) *A* has fractional Hausdorff dimension, and (2) (M, Φ_t) is chaotic when restricted to *A*.

Note that in our ongoing discussion on dynamic systems, no section is directly dependent on the system being deterministic or stochastic, so long as the derivatives and time differentials involved exist. Chaos can arise from simple linear deterministic systems or be absent from complex non-linear stochastic systems. Chaotic games can then be formed from a dynamic game that involves a chaotic dynamic system, independent of the payoff constraint structure. Here we may then define a general fractal game.

Def. A game Γ_f is a *fractal game* if its dynamics are controlled by a (stochastic or deterministic) dynamic system (M, H, P, C) in which a strange attractor *A* exists.

If Φ_t^μ can be parameterized by μ such that the properties of (M, Φ_t^μ) change dramatically at some value $\mu = \mu_0$, say stable to unstable, typical to chaotic, change structures entirely, or merge into a combination of structures, then the behavior of (M, Φ_t^μ) is said to *bifurcate* at $\mu = \mu_0$, a *bifurcation point* of (M, Φ_t^μ). Ergodic properties of (M, Φ_t^μ) may also be investigated. A dynamic system defined on a probability space

$M_\lambda = (M, \Sigma, \lambda)$, is said to be *ergodic* if the time average and space average are equal with respect to ergodic transformations T on M_λ. T is an *ergodic transformation* on (M_λ, Φ^t) if $A \in \Sigma$ and $\lambda(T^{-1}(A)) = \lambda(A)$ imply $\lambda(A) = 0$ or 1.

Formally, this means:

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{k=1}^{n}\Phi_t\left(T^k(x)\right) = \int_x \Phi_t(x)d\lambda .$$

We can define a corresponding operator, $U^t : \mathbb{R} \to \mathbb{R}$ and maps *a*, called observables that assign physical attributes (numbers) to points in the phase space, such that $U^t a(x) = a\Phi^{-t}(x)$. This maps a non-linear finite-dimensional dynamic system (M_λ, Φ^t) to an infinite-dimensional linear system $\left(\mathbb{R}, U^t\right)$. Ergodic properties of (M_λ, Φ^t) can then be studied from the spectral properties of U^t. In particular, it has been shown that there exists an absolutely continuous measure λ_{SRB} defined on a probability space, $M_\lambda = (M, \Sigma, \mu)$ such that in a chaotic dynamic system, (M_λ, Φ^t), restricted to an attractor *A*, is ergodic (Ruelle, 1976). Chaotic games defined on a probability space M_λ will then exhibit ergodicity in attractors that have non-empty intersection with the set $\left\{x; C\left[P\left(\Phi_t(x)\right)\right] \text{ is true}\right\}$.

Fuzzy Games

Fuzzy games are modeled using the notions of fuzzy sets introduced by Zadeh (1965). Fuzzy games were originally developed by Butnariu, with later revisions and modifications by Billot (Butnariu, 1978) (Billot, 1992). In Butnariu-Billot models, fuzziness is introduced through the beliefs or attitudes that an agent may possess about the actions or strategies of other agents. This *psychological game* can be modeled through constraints imposed on the strategy spaces available to the agents. Equilibriums may exist when common beliefs among the agents are shared. Fuzzy games in which both payoffs and strategies are fuzzy were developed for noncooperative settings by Aristodou and Sarangi in an attempt to generalize the decision framework of Bellman and Zadeh (Aristodou & Sarangi, 2005). An *n*-person non-cooperative fuzzy game in normal form in which only strategy space is fuzzified will be defined first.

Def. Let $\Gamma = (S_i, W_i, \Pi_i)_{i=1}^n$ where the set of agents is represented by $I = \{1,..., n\}$ and for each agent *i*:

(1) S_i is the set of pure strategies available,

(2) $\omega_i = (\omega_1^i,..., \omega_n^i) \in W_i$ are weights assigned such that the weight ω_i^m is agent *i*'s preference for the *m-th* pure strategy in S_i ,a *strategic arrangement*. The vector $\omega = (\omega_1 ,..., \omega_n) \in W = \underset{i \in I}{\times} W_i$ is called a *strategic choice* for the game Γ ,

(3) $\Pi_i \in 2^W$ and for all $\omega \in W$, $\pi_i(\omega)$ is the possibility assigned by agent *i* to the strategic choice ω. $\pi_i(\omega)$ is also the membership function that assigns a membership value to each mixed strategy utilized by agent *i*, and

(4) letting $W_{-i} = \underset{j \neq i}{\times} W_j$ and $W_i = 2^{W_{-i}} \times W_i$, $s_i = (A_f^i, w^i) \in W_i$ is agent *i*'s *strategic concept* in Γ, where $A_f^i = \{x \in X : \mu_{A^i}(x) \geq f\}$ is the usual α - *cut* of the fuzzy set A^i (Saranji, 2000, p.101).

In addition to the above four conditions, Γ should also satisfy the fifth following condition:

(5) if $A_f^i \in 2^{Y_{-i}}$ and $A_f^i \neq 0$, then $\pi_i(A_f^i) \neq 0$.

This condition means that there exists a strategy $s_i \in W_i$ such that $\pi_i(A_f^i)(w^i) \neq 0$, i.e., each agent possesses, at least, one non-trivial strategy to play in the game.

Def. The vector $s = (s_1,..., s_n)$ is called a *play* in Γ. Denote the set of all possible plays in Γ by *S*.

Def. For two strategies available to one agent, $s_i^1, s_i^2 \in S_i$, s_i^1 is strictly *better* or is a strictly *better strategic conception* than s_i^2 for agent *i* if and only if $\pi_i(A_f^1)(w^1) > \pi_i(A_f^2)(w^2)$. We denote this by $s_i^1 \succ s_i^2$.

This is a fuzzy version of dominance within the class of strategies for an agent in a classic game.

Def. For dominance of plays or strategy profiles, we say that for two plays $s^1, s^2 \in S$, s^1 is *socially preferable* to s_i^2 if and only if for all $i \in I$, $\pi_i(A_f^1)(w^1) > \pi_i(A_f^2)(w^2)$. We denote this by $s^1 \succ s^2$.

Def. A *possible solution s*, of the game Γ satisfies the following: for any other play s^*, the condition $s_i^* \succ s$ cannot be true, that is, for all $i \in I$, $\pi_i(A_f^i)(w^i) > \pi_i(A_f^*)(w^*)$.

Def. A play $s^* = (A_f^*, w^*)$ is called a *play with perfect information* when it is of the form:

$$(A_f^*)(w^{1*},..., \omega^{(i-1)*}, \omega^{(i+1)*},..., \omega^{n*}) = 1, \text{ if } \omega^j = \omega^{j*} \text{ where } j \neq i$$
$$0, \text{ otherwise}$$

An alternative and equivalent condition to this is that $\omega^{-i*} \in A_f^{i*}$ for all $i \in I$, the *mutual consistency condition on beliefs* on all agents.

Def. An *equilibrium* strategy of the game Γ is a possible solution $s = (A_f^{i*}, \omega^{i*})$ that satisfies the mutual consistency condition on beliefs on all agents $i \in I$.

Now consider an *n*-person non-cooperative fuzzy game in which both payoffs and strategies are fuzzified. We follow the work of Aristodou & Sarangi (Aristodou & Sarangi, 2005). We frame a game with fuzzy strategies and payoffs as:

Def. Let $\Gamma^f = (S, \Pi, \mu, \gamma)$ where the set of agents is represented by $I = \{1,..., n\}$ and for each agent *i*: S_i is its strategy space and $\Pi_i : S \to \mathbb{R}$ is its payoff function. Here $S = \bigotimes_{i=1}^{n} S_i$ is the space of all strategy profiles. $\mu_i : S_i \to [0,1]$ defines an agent's "perception constraint", i.e., fuzzy belief constraint on the strategies of others and hence a constraint on their own strategies. $\gamma_i : S_i \to [0,1]$ defines an agent's "aspiration level", i.e., fuzzy goal function that can capture the agent's alternative to utility maximization such as in altruistic behavior.

Def. Agent *i*'s *decision set* is given by $\delta_i(s) = \min\{\mu_i(s_i), \gamma_i(s_i)\}$, i.e., is the intersection of the set of goals and the constraints.

Def. A strategy tuple $s^* = (s_1^*, ..., s_n^*)$ is a *Nash equilibrium* in Γ^f if for all $i \in I$, we have $\delta_i(s^*) \geq \delta_i(s_i^{'}, s_{-i}^*)$ for all $s_i^{'} \in S_i$.

In the proceeding discussion, we assume that S_i is compact and convex and the payoffs are continuous for all $i \in I$.

Theorem. In a fuzzy game, $\Gamma^f = (S, \Pi, \mu, \gamma)$, if δ_i is non-empty, continuous and strictly quasi-convex in an agent's own strategies, then Γ^f has at least one Nash equilibrium.

Proposition. Let $C_i = \max\limits_{s_i \in S_i} \min\limits_{s_j \in S_j} \delta_i(s) = \max\limits_{s_i \in S_i} \min \left\{\mu_{\text{i}}(s_i), \min\limits_{s_j \in S_j} \gamma_i(s_j)\right\}$, with the α-cut of S_i defined as $S_i^{\gamma^0} = \{s_i \in S_i : \mu_i(s_i) \geq \gamma_0\}$ and $S_i(\gamma_0) = \left\{s_i \in S_i : \min\limits_{s_j \in S_j} \gamma_i(s) \geq \gamma_0\right\} \subseteq S_i$. Then if $S_i(\gamma_0) \neq 0$ and $S_i(\gamma_0) \cap S_i^{\gamma^0} \neq 0$, then $C_i \geq \gamma_0$.

Corollary. Let Γ^f be a one-sum game, i.e., $\delta_i(s) = 1 - \delta_j(s)$ for $i \neq j$. Let δ_i be non-empty, continuous, and strictly quasi-concave in an agent's own strategies. If $C_i \geq \gamma_0$, then it is also a Nash equilibrium of Γ^f.

In pre-play communication agents may confer before the start of the fuzzy game. In this case, if this communication leads to a better knowledge of what strategies to use, the pre-play strategies will be contained in the non-pre-play strategies of an agent, i.e., $S_i^{'} \subset S_i$, where $S_i^{'}$ is the pre-play strategy set for agent *i*. By the definition of C_i, $C_i^{'} \geq C_i$ and hence a better possibility for coordination. This is not generally true in a classical game.

Quantum Games

The revolution that was started by the advent of Quantum Mechanics in describing the probabilistic behavior of sub-particles in the universe, also gave a new meaning to the concepts of information and entropy. Game theory uses the concepts of information theory in order to describe distances between targeted strategies, i.e., good strategies and sub-optimal strategies. Payoff functions can be described by information distance metrics that measure the reward or penalty for reaching certain plateaus in performance. Mixed or randomized strategies can be viewed from the lens of quantum super-positions of strategies. When one agent chooses a strategy, it is effectively communicating information to the rest of the agents of the game. This information is at its root, quantum in nature. So, quantum entanglement may connect one agent's choice of stratagem with all others, in a non-trivial fashion. Already, games have been devised wherein a spin-flipping game (a quantum variant of the penny flipping game), a quantum strategy will always beat the best classical strategies (Meyer, 1999). It was recently conjectured that in a non-zero sum game, quantum strategies can be constructed that will always give rewards when played against any classical strategy (Eisert, Wilkens, Lewenstein, 1999). However, it was pointed out in a comment to this claim that an agent must have its strategy choice mechanism extended to include picking probabilistic choices (Benjamin & Hayden, 2000), rather than just completely positive maps. Quantum entanglement can also be introduced through the payoff functions. We shall set up the quantization of games in order that quantum mechanisms be applicable to game elements.

In a version of a quantum game, *qubits* represent agents' information containers. This is in contrast to or more succinctly, a generalization of the *0-1* logic of the computational bit. Let $\Gamma = (I, S, U)$ be a game that will be quantized for a quantum version. A quantum game can then be realized from Γ in the following way:

Let $\mathcal{H}$ depict a Hilbert space representing the states of Γ, $\rho \in \mathcal{H}$ is an initial state, and S is now the space of permissible quantum operators (unitary) of the *n* agents. Then the classical game can be rewritten as $\Gamma = (I, \mathcal{H}, S, U)$. S will be a space of completely positive trace-preserving maps from $\mathcal{H}$ to itself, i.e., if $s \in S$, then *s* satisfies the following condition: if $tr(\rho) = 1$, then $tr[s(\rho)] = 1$ and is $\rho > 0$ is a positive operator, then $s(\rho) > 0$. Strategies are now written in the quantum mechanics' *ket* notation, $|\psi>_k \in H_k$ where H_k is a Hilbert space of permissible quantum operators for agent *k*. The state of the game can then be written as the sum of quantum operators acting as agent strategies, $|\Psi>_{in} = \sum_k |\psi>_k$. Strategy operators that are "fair" are symmetric with respect to the interchange of agents. S may also be thought of as a subset of the group of unitary operators acting on an *n*-dimensional vector space, denoted by $U(n)$. Let $s = (s_1,..., s_n)$ depict a strategy profile acted on by the agents initially. In a quantum scenario, each agent has two physical devices, one for the manipulation of the agent's qubit, $\Im$, and one for the measurement device which will determine the payoff from the state of the *n* qubits, $\Im^t$. Initially, there is a source of the qubits for each agent. The final state of the game can be computed as,

$$|\psi_f >= \Im^t (s_1 \otimes ... \otimes s_n) \Im | s \tag{2.15}$$

The expected payoff for agent i can be given as the linear combination:

$$u_i(s) = \sum_r \lambda_r \left| \left\langle r | \psi_f \right\rangle \right|^2 \tag{2.16}$$

where $\sum_r \lambda_r = 1$, $\lambda_r \geq 0$ for each *r*, and the sums are taken over all feasible n-tuples of $\underset{i=1,\ldots,n}{\times} \mathcal{H}_i$. Payoffs are calculated similarly for other agents by interchanging the λ_r .

Generalizing the results of Benjamin and Hayden (Benjamin & Hayden, 2000), we construct a quantum game scenario that depicts a Nash equilibrium. Suppose agent *i* chooses a (unitary operator on *n*-dim vector spaces) strategy $s_i \in U(n)$ randomly with respect to a suitable Haar measure on $U(n)$. Let the competing coalition of agents denoted by -*i* respond by choosing a strategy, $s_{-i} \in \underset{j=1,\ldots,n-1}{\times} U(n) = U^{n-1}(n)$. Then the probability of the outcome of a state consisting of some feasible n-tuple, $\sigma = (\sigma_1, \ldots, \sigma_n)$ is:

$$\begin{aligned} p_\sigma(s_{-i}) &= \int_{U^{n-1}(n)} \left| \left\langle \sigma \mid \Im^t (s_1 \otimes \ldots \otimes s\ )\Im \mid \sigma_0 \right\rangle \right| dx_{-i} \\ &= \int_{U^{n-1}(n)} \left| \left\langle \sigma \mid \Im^t (s_1 s_0^t s_0 \otimes I)\Im \mid \sigma_0 \right\rangle \right| dx_{-i} \\ &= \int_{U^{n-1}(n)} \left| \left\langle \sigma \mid \Im^t (s_1 \otimes I)\Im \mid \sigma_0 \right\rangle \right| dx_{-i} \\ &= p_\sigma(I), \end{aligned}$$

where $s_i \in U(n)$ was chosen so that $(s_i \otimes I)\Im \mid \sigma_0 = (I \otimes s_{-i})\Im \mid \sigma_0$, σ_0 is the pure strategy depicting the existence of agent *i*'s information only, utilizing the right invariance of the Haar measure, normalized so that $volume[U^{n-1}(n)] = 1$. Then the choice of strategies by the coalition -*i* will not matter because their respective payoff will be the average of all the classical payoffs. When agents choose as above, using this random strategy, a Nash equilibrium is reached because the agents cannot improve their respective payoffs by simultaneously changing stratagem. Higher dimensional quantum computation of states may give equilibrium strategies that are below or above classical equilibrium payoffs, and above classical cooperative equilibrium payoff are on regular basis (Benjamin & Hayden, 2000). This aspect of *n*-person quantum games should be researched further. In this regard, if quantum computation cannot always give a better equilibrium strategy than the classical one, it can be used to give a better cooperative equilibrium strategy. What this means is that quantum strategies will elevate the cooperative payoffs of classical games, i.e., give an overall incentive for the voluntary conversion of all feasible games to cooperative games.

Recently, in Brunner and Linden (2013), by expressing Bayesian games in which a correspondence is made between game components and a test for Bell inequilities in a quantum two qubit system (using the usual Alice-Bob observer duality with a single source of classical particles being measured), quantum game strategies are proposed to be superior to classical strategies. The correspondence used is the following: the source of particles for the test is equated with a common game advisor giving players information leading to correlated strategies in a game, the observers Alice and Bob are the game agents, and the average payoff function (for each agent) for the Bayesian game is a Bell expression. In a general N-agent situation, one can express an average payoff for an agent in a Bayesian game $\Gamma = \left(N, \Omega, \mu, \prod_{i\in\{1,\dots,N\}} \mathcal{A}_i, \prod_{i\in\{1,\dots,N\}} \chi_i, (\tau_i)_{i\in\{1,\dots,N\}}, (v_i)_{i\in\{1,\dots,N\}} \right)$ by,

$$V_i = \sum_{(X_i)_{i\in I_N} \in \prod_{ii\in In} X_i, (A_i)_{i\in I_N} \in \prod_{ii\in I} \mathcal{A}_i} \mu\left((X_i)_{i\in I_N} \right) v_i\left((X_i)_{i\in I_N}, (A_i)_{i\in I_N} \right) p\left((A_i)_{i\in I_N} \mid (X_i)_{i\in I_N} \right) \qquad (4.44)$$

where $I_n = \{1,\dots,N\}$ is the index space, N is the number of agents, Ω is the universe of discourse (states) for Nature, μ is a prior probability distribution on Ω , $\prod_{i\in I_n} \mathcal{A}_i$ is the product space of actions, $(\chi_i)_{i\in I_n}$ is the product space of agent types, $(\tau_i)_{i\in I_n}, \tau_i : \Omega \to \chi_i$ a product space of mappings that assign a state of nature to an agent type,

$\left(v_i\right)_{i\in I_n}, v_i : \Omega \times \left(\prod_{i\in I_n} \mathcal{A}_i\right) \to \mathcal{R}$ are the payoff functions. In a Bayesian game, the advisor that correlates information (the source of particles to be measured by the *N* agents) to the *N* agents is not privy to the state of Nature. Correlated equilibria in Γ are then pursued. Agents are asked questions which correspond to agent types and are answered with answers which correspond to actions. The joint probability distribution, $p\left(\left(A_i\right)_{i\in I_n} \mid \left(X_i\right)_{i\in I_n}\right)$ is approximated by an appropriate statistic of repeated tests of questions and answers to agents. Bell inequalities are given by linear expressions of those joint probabilities,

$$S\left(p\right) = \sum \alpha_{\left(X_i\right)_{i\in I_n}, \left(A_i\right)_{i\in I_n}} p\left(\left(A_i\right)_{i\in I_n} \mid \left(X_i\right)_{i\in I_n}\right) \tag{4.45}$$

where $\alpha_{\left(X_i\right)_{i\in I_n}, \left(A_i\right)_{i\in I_n}} \in \mathcal{R}$. The probabilities $p\left(\left(A_i\right)_{i\in I_N} \mid \left(X_i\right)_{i\in I_N}\right)$ are the possible strategies. Let $L = \max_{p\in\mathcal{P}\left(D_N\right)} S\left(p\right)$, where the probabilities *p* are taken over the space of all possible probabilities on $\mathcal{D}_N = \left(\prod_{i\in I_N} X_i, \prod_{i\in I_N} \mathcal{A}_i\right)$, given by $\mathcal{P}\left(D_N\right)$. Bell inequalities are then of the form $S\left(p\right) \le L$. Then Bayesian game average payoff functions given by (4.44) are Bell expressions (4.45) with the correspondence $\mu\left(\left(X_i\right)_{i\in I_N}\right) v_i\left(\left(X_i\right)_{i\in I_N}, \left(A_i\right)_{i\in I_N}\right) = \alpha_{\left(X_i\right)_{i\in I_N}, \left(A_i\right)_{i\in I_N}}$. Using the classical local bounds of a Bell inequality, agent average payoffs must satisfy $V_i \le L$.

Now consider the convex *n*-polytope which defines the possible space of points having components that are the agent average payoffs. Linear combinations of payoffs can also be used to get conditions, $\sum_{i \in \{1,\ldots,N\}} \beta_i V_i \leq \beta_0$. These inequalities are then facets of the *n*-polytope and the Bell inequalities $S = \sum_{i \in I_N} \beta_i V_i$ using the local bound $L = \beta_0$ can then be used analogously. Average payoffs are then Bell expressions and are limited by Bells inequalities when a classical source is used. It is well known that in quantum correlation settings, Bell inequalities can be violated. Non-local super-quantum correlations have been shown to exist which satisfy the no-signaling principle, communicating in sub-luminal speed, while possessing stronger than quantum correlations. In a quantum Bayesian game with non-local correlations using a quantum advisor source (entanglement case). Bell inequalities in such games are then violated and as such super-quantum strategies in quantum Bayesian games outperform both quantum and classical strategies. Entangled systems using entangled bits or what we call *e-bits* in correlated quantum games are then a way in which to obtain what we call *super-correlated strategies*. In our concept of inception games, injecting entanglement using super-quantum correlated stochastic (recursiveness), implies the existence of super-correlated strategies for guru-consciousness and hence social dominance. The expressiveness of using GTU constraints for super-quantum correlated gravity systems with generalized probability causation across spacetime fabrics further enhances the generality of very powerful social systems.

Lipschitz Games

Global properties of games are usually described through the existence of final payoffs and solutions through equilibria as reviewed and discussed before. Nonetheless, local agent influence on the game components of other agents is of paramount importance because of perturbative effects, if games are viewed as networks of agents with edges representing transitions to other states in terms of payoff potentials. The concept of Lipschitz games describes the effect that a single agent's change of strategies will have on other agents in a normal form game. In a Lipschitz game, a Lipschitz constant is the maximal amount of change in an agent that a change in strategy of another agent will cause. More detailed, we review the space of Lipschitz games $L(n,m,\Delta)$, as those normal form games *G* with Lipschitz constant $\delta(G)>0$ less than a given amount $\Delta>0$, with *n* agents with at most *m* strategies each.

Def. The *Lipschitz constant* of a normal form game $G=(\mathrm{I},A,P),|I|=n,A=\prod_{i\in I}A_i$, where *I* is the agent index set with cardinality *n*, *A* is the strategy profile space for all *n* agents, with payoffs $\{L_i\}_{i\in I}, L_i : A\to\mathcal{R}$, is the following constant

$$\delta(G)=\max_{i\in I, a_i\in A_i, \forall\left(a'_{-i},a''_{-i}\right)\ni\rho\left(a'_{-i},a''_{-i}\right)=1}\left\{\left|L_i\left(a_i,a'_i\right)-L_i\left(a_i,a''_i\right)\right|\right\} \tag{4.46}$$

where a metric on opponent spaces of actions $A_{-i}=\prod_{j\neq i}A_j$ is given by

$\rho\left(a'_{-i},a''_{-i}\right)=\#\left\{1\leq j\leq n \mid j\neq i, a'_j\neq a''_j\right\}$.

The metric ρ on A_{-i} for each *i*, measures the number of occurrences where agent actions are different for two agent strategy profiles in A_{-i}. The main result for Lipschitz games is: for $\varepsilon > 0$ and $m, n \in \mathcal{N}$, every game $G \in L(n, m, \Delta)$ with $\Delta = \frac{\varepsilon}{\sqrt{8n \log(2mn)}}$ admits a pure ε-equilibrium (Azrieli and Shmaya, 2013). Lipschitz games are connected to the concepts of continuity and anonymity. Continuity refers to an agent's behavior having minimal impact or effect on the payoffs of other agents. Whereas anonymity refers to the phenomena where each agent's payoff depends on a notion of a collective or aggregate behavior of other agents. Lipschitz games then define a spectrum (using its metric) for defining degrees of anonymity and continuity piecewise. In inception games, defining Lipschitz constants measures the propensity of agents to cause defections of others or changing their own social power or influence and therefore, the propensity to change an inception potential. Classes of inceptions may then be partitioned into Lipschitz-bounded groups of inception games, (i.e., inceptions in Lipschitz-bounded game classes are influentially bounded or can be measured to be a distance from certain defection potentials). A Lipschitz-bounded (LB) class of games can be defined as

$$L(m, n, \delta_2, \delta_1) = L(m, n, \delta_2) \setminus L(m, n, \delta_1), \delta_2 > \delta_1 \tag{4.47}$$

Inception games *G* may then be subdivided into linked LB subgames in $L(m, n, \delta_2, \delta_1)$ for appropriate subranges of *m* and *n*. Agents participating in subgames $G_{(\delta_1, \delta_2)} \in L(m, n, \delta_2, \delta_1)$ are then influence-bounded (social order-bounded) by the interval

$[\delta_1, \delta_2]$ and the LB subgame inception will admit an ε-pure equilibrium for

$$\delta_1 \le \frac{\varepsilon}{\sqrt{8n\log(2mn)}} \le \delta_2 \ .$$

Utility Theory Revisited

Game structures depend on the mechanism of utility or an equivalent concept. Of the utility measures reviewed and discussed in the breadth section, all were, in some way dependent on a subjective or heuristic measure of relevance towards a judgment or probability of occurrence of an event or circumstance. Suppose one relaxes the dependency on absolute probabilities when calculating utility using a consistent method of judgment and validation. In this section, two separate general approaches will be reviewed and presented that endeavor to do exactly that. The first is a methodology that proposes to solve multi-criteria decision problems with subjective judgments that do not involve relative scales from probabilities or other subjective calculations, the Analytic Network Process and its parent, the Analytic Hierarchy Process. The second is the category of methods that utilize belief functions, either applied to probability structures or taken alone. These methodologies are (1) Upper and Lower Probabilities (ULP), (2) Dempster-Shafer models of evidentiary value (EVM), (3) probability of model propositions, and (4) the transferable belief model (TBM). The first three methods are generalizations of the Bayesian probability approach, while the fourth is based solely on belief functions.

The Analytic Hierarchy Process (ATP) and its predecessor, the Analytic Network

Process (ANP) are methods developed by Saaty to calculate the consistency of possible judgments and hence, of actions taken in assessing an outcome (Saaty, 1996). These are methods for the solution to multi-criteria decision-making. Instead of utilizing prior probability structures and calculating utility based on them, the ATP and ANP use judgment measures based on an absolute scale, assigned to each strategy. These judgments are dominance factors of one strategy against another in pair-wise fashion, i.e., how more dominant is one element over another with respect to a criteria, attribute, or goal? The scale is from the integers in $\{1,...,9\}$. This is based on the acuity of humans to identify with the decimal system as unambiguously as possible. It may be that this is a bias toward western phenomena since different number systems exist in other cultures. A dominance score of $a_{ij} = 1$ for strategy s_i against strategy s_j in regards to an objective means that both contribute equally. If $a_{ij} = 9$ then s_i is 9 times as strong as s_j in obtaining the objective. Now let $w = (w_1,...,w_n)$ be the vector of unknown scaled influences for each of *n* objects or strategies to be compared. A matrix, $W = (w_{ij}) = \left(\frac{w_i}{w_j}\right)$, is then formed. If all pair-wise judgments are consistent then $w_{ji} = \frac{1}{w_{ij}}$. In the hypothetical *W*, the matrix is consistent. In a consistent matrix, all diagonal entries are 1. Furthermore, the principal eigenvalue of *W* is *n*, where *n* is the number of objects being compared. The corresponding eigenvector would be $w = (w_1, w_2, ..., w_n)$, the original scaled judgments. In a consistent matrix, one has the following property: $w_{ij} w_{jk} = w_{ik}$ for all $i, j, k = 1,...,n$.

Hence, $W^k = n^{k-1}W$. Therefore, in normalized form, W and W^k have the same principal eigenvector. In a general matrix, this may not be true, i.e., in an inconsistent matrix. Finding the principal eigenvector of a general tabulated judgment matrix A, gives a way to recapture the original scaled individual judgment vector. So, if one surveys a group for pair-wise judgments between a set of *n* strategies, $A = \left(a_{ij}\right)_{n,n}$, where a_{ij} is the judgment comparison of strategy *i* versus strategy *j*, the principal eigenvector of *A* will give an estimate of the original unknown individual scaled judgments $w = (w_1, w_2, ..., w_n)$. The principal eigenvector can be transformed into a normalized or distributive form by dividing each element in it by the sum of all it elements, i.e., $w_D = \frac{1}{a}\left(w_1, w_2, ..., w_n\right)$, where $a = \sum_{i=1}^{n} w_i$. It can also be transformed into an ideal form where each element is divided by the maximum element in the vector, i.e., $w_I = \frac{1}{m}\left(w_1, w_2, ..., w_n\right)$, where $m = \max_{1 \le i \le n} w_i$. For a given judgment matrix $A = \left(a_{ij}\right)_{n,n}$, the priority measure for the strategy *i*, p_i , can be computed as $p_i = \frac{a_{1i}}{\sum_{j=1}^{n} a_{1j}} = \frac{\sum_{j=1}^{n} a_{1j}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}}$. The total priority, p_i^T , for the strategy *i* is then just the normalized priority across all strategies, i.e., $p_i^T = \frac{p_i}{\sum_{j=1}^{n} p_j}$. Now let $\lambda_{\max}$ be the computed principal eigenvalue of *A*. Define a consistency index for *A* as:

$$\mu = \frac{\lambda_{\max} - n}{n-1} \tag{2.19}$$

This is a relative measure of the difference in eigenvalues between the judgment matrix and a hypothetical and exact priority matrix. An inconsistent judgment matrix would then be one that is exceedingly large. Saaty suggests a value of approximately .10 as a threshold reasoning that some inconsistency is needed in order to learn from new data, whereas large inconsistencies mean that one is not grounded experientially. The value of .10 translates into the property of inconsistency possessing one order of magnitude lower in importance than consistency, i.e. one values being consistent with observed data 9 times more than deviating from this conformance.

The AHP model assumes that elements are independent of each other, hence pair-wise comparisons separately, and that no feedback occurs from dynamic changes that may occur doing interactions. The ANP was developed to remedy this. First, besides forming the pair-wise judgments, for each pair of strategies, a third influence is calculated relative to a third strategy, which of the two strategies in the pair Influence the third strategy more. This acts to cover the interaction that may involve multiple comparisons. The second alteration is in the formation of the matrix of judgments. Since multiple influences will be compared the matrix can now be separated into cluster matrices that represent the various components of a general network system. This system does not have to be a hierarchy as in the AHP. The judgment matrix will then form a supermatrix whose elements are comprises of component judgment matrices. In a system let C_h represent the *h-th* component out of *N*. Let n_h be the number of elements in that component. We denote these n_h elements by $\left(e_{h1},...,e_{hn_h}\right)$. Form a super-matrix W as:

$$W = \begin{bmatrix} W_{11} & ... & ... & W_{1N} \\ . & W_{22} & & . \\ . & ... & ... & . \\ W_{N1} & ... & ... & W_{NN} \end{bmatrix}, \text{ where } W_{ij} = \begin{bmatrix} w_{i1}^{j1} & ... & ... & w_{i1}^{jn_j} \\ . & w_{i2}^{j2} & & . \\ . & ... & ... & . \\ w_{in_i}^{j1} & ... & ... & w_{in_i}^{jn_j} \end{bmatrix} \tag{2.20}$$

For each criteria or control, C, a judgment super-matrix W_C is computed. Priorities are computed as before except that now second comparison in relation to a third element is also done. The super-matrix is turned into a column stochastic matrix, ${}^{s}W_C$, one whose columns add up to unity by this normalization of weights. Next the limiting sequence of powers, ${}^{S}W_C^{\infty} = \lim_{k \to \infty} {}^{s}W_C^{k}$ is investigated. The limiting priorities of ${}^{S}W_C^{\infty}$, a stochastic matrix, depend on its reducibility, primitivity, and cyclicity. Saaty derives results for all four possible cases using Sylvester's theorem on the form of a function of a matrix. The function, in this case, is the limit as the powers tend to ∞. Consistency indexes can then be calculated based on the closed form calculation of ${}^{S}W_C^{\infty}$ and its priorities. If consistencies are insufficient, the DM may modify the most inaccurate of judgments, iterating as such. In this way, a method is derived that can assist in clarifying what a consistent and effective decision or strategy is in a multi-criteria decision problem. No prior probabilities were used, albeit, an iterative series of subjective judgments based on an absolute scale are needed as a starting point.

The group of models labeled as Dempster-Shafer Theory cover several criteria for judging decisions. These models structure the degrees of belief about a judgment be it from probabilities or otherwise. Beliefs result from an inherent uncertainty about information. Fuzzy and Possibility Theories cover the cases of vagueness and ambiguity. The transferable belief model (TBM), will be used as a meta-model for models that utilize belief functions since they are generalizations of Dempster models, Bayesian probability, and other non-probabilistic models. We will closely follow the work from Smets (Smets, 2000)

The common setup for belief function-based approaches is the following: Let Ω depict a finite set of possible worlds of discourse. Let ω_0 represent the actual world in our existence. Denote by $bel(A)$ a "measure of strength" in the belief an agent has that $\omega_0 \in A \subset \Omega$. Beliefs will satisfy three conditions,

(1) $bel(A) \leq bel(B)$ or $bel(A) \geq bel(B)$,

(2) if $A \subseteq B \subseteq \Omega$ then $bel(A) \leq bel(B)$, and

(3) $bel: 2^{\Omega} \rightarrow [0,1]$ (or any other finite interval in $\mathbb{R}^{+}$)

Every belief-based model should have two components, (1) a static part in which a description of the state of the belief exists given information available to an agent, and (2) a dynamic part, a mechanism for updating that belief when new information arrives for the agent. There are two ways that beliefs may be used by an agent, one is credal, beliefs are accepted as, and pignistic, where beliefs are utilized to make decisions. These levels are philosophically different and hence, the assumption for using probability to justify one from the other is not required. Belief functions derived from the credal level will lead to the pignistic methods for making decisions. The TBM justifies beliefs. TBM generalizes Bayesian probability because it is based on beliefs, a generalization to probabilities. Beliefs are super-additive as probabilities are additive, i.e., $bel(A \cup B) \geq bel(A) + bel(B) - bel(A \cap B)$. In the TBM, beliefs satisfy the following monotone condition:

$$bel(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} bel(A_i) - \sum_{i>j} bel(A_i \bigcap A_j) \ldots - (-1)^n bel(\bigcap_{j=1}^{n} A_j)$$

Define the *basic belief assignment* (bba) on a set *A*, as a mapping $m: 2^{\Omega} \to [0,1]$ that satisfies $\sum_{A \subseteq \Omega} m(A) = 1$, i.e., $m(A)$ is the "most specific belief" that $\omega_0 \in A$. Put another way, if $m(A) > 0$ then $m(B) = 0 \ \forall B \subseteq A$ and $B \neq A$. Define a general belief function using bba's as:

$$bel(\varnothing) = 0 \text{ and } bel(A) = \sum_{\varnothing \neq B \subseteq A} m(B) \quad \forall A \subseteq \Omega, A \neq \varnothing.$$

Note that when $m(A) = 0 \ \ \forall A \subseteq \Omega$ with $|A| \neq 1$ *bel* becomes a probability function and the TBM reduces to the Bayesian Theory. Now define the dual of belief, the *plausibility function, pl*:

$$pl(A) = bel(\Omega) - bel(\overline{A}) = \sum_{X \subseteq \Omega,\ X \bigcap A \neq 0} m(X), \quad \forall A \subseteq \Omega$$

pl(A) gives a kind of generalized global measure of support in Ω that $\omega_0 \in A$. Now define two functions to be used in the conditioning combining of beliefs, the *commonality* and *implicability* functions, respectively, $q, b: 2^{\Omega} \to [0,1]$:

$$q(A) = \sum_{X \subseteq \Omega, A \subseteq X} m(X), \quad \forall A \subseteq \Omega$$
$$b(A) = bel(A) + m(\varnothing) = \sum_{X \subseteq \Omega, X \subseteq A} m(X), \quad \forall A \subseteq \Omega$$

Define the bba and belief functions conditioned on a set *A* as:

$$m_A(B) = \sum_{C \subseteq \overline{A}} m(B \cap C)$$
$$bel_A(B) = bel(B \cup \overline{A}) - bel(\overline{A}), \quad pl_A(B) = pl(B \cap A), \quad b_A(B) = b(B \cap \overline{A})$$
$$q_A(B) = \begin{cases} q(B) & \text{if } B \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

These are Dempster's rules of conditioning. In order to continue to the pignistic reason for beliefs, one must construct a proxy for a probability function to be used in decision-making. This probability function must come from our construct of beliefs from the credal stage. To this end, we assume that a probability function will be a function of the belief function. The transformed probability function built on *bel* will be denoted by P_{bel}. The transformation mapping doing this will be denoted by $\Gamma(bel, F)$. Then $P_{bet} = \Gamma(bel, F)$. It will be dependent on both the belief function and the betting frame, *F*. The betting frame is the set of atoms in Ω which will be the objects betted on. Wages are bet on the sets in 2^F only. Bets are then assigned to atoms independently. Granules in *F* are defined as sets of atoms that have been assigned equal bets. *F* is then built using refinements and coarsenings of granules from an initial frame, $F_0 \subseteq \Omega$, that was based on an initial credibility function. P_{bet} is then called a pignistic probability and is a classical probability measure. The pignistic probability can be given a precise definition:

$$P_{bel}(\omega) = \sum_{A, w \in A \subseteq \Omega} \frac{m(A)}{|A|(1 - m(\varnothing))}$$

The philosophy of the upper and lower probability model (ULP) is that a belief structure cannot be defined by a single function when information is missing about prior wisdom. This translates into bounding all compatible beliefs by a constraint defined by a set of probability measures. IN actuality, this is done by finding the sup and inf of *P(A)* where *P* is a member of a constraining set of probability measures, π. Generally, any constraint law can be used using the probability measures from the constraint set of probability measures, π, for example, convex combinations from π or particular types of measures, such as discrete or heavy or light tailed measures.

Hybrid Games

Each of the explored structures of games and utility measurement reviewed had common features; the involvement of multiple agents (one of which may be nature), strategy or decision spaces for those agents, and payoff measurements for using those strategies. One may be better served by utilizing a more abstract notion of a game if one is to build a general holistic framework of games involving multiple aspects of dynamicism, behavior of organisms and physical entities, and uncertainty with or without resulting chaotic patterns.

We start by using the concept of an abstract game or economy, which generalizes the structure of preference for strategies from agents. This richer structure will be a better container for the varied approaches that we have reviewed.

Definition An *abstract economy* or *abstract game* will be a triplet

$$\Gamma = (I; (S_i)_{i \in I}, (P_i)_{i \in I}, (F_i)_{i \in I})$$

where

(1) I is a non-empty set or index of agents,

(2) S_i is a non-empty strategy space containing the feasible strategies for agent *i*,

(3) $P_i : S \Rightarrow S$ is a strict preference relation on S, for agent *i*, and

(4) $F_i : S \Rightarrow S_i$ is the constraint relation for agent *i*.

F_i will restrict which strategies are feasible for agent *i*, given the strategy choices for the agent coalition -*i*. If $F_i(s) = S_i$ for all *i* and *s*, then the game is a regular one as reviewed above.

Def. For each $i \in I$, a *good reply* is a relation $U_i : S \Rightarrow S_i$, defined by $U_i(s) = \{s_i^* \in S_i : (s_i^*, s_{-i}) \in P_i(s)\}$.

Def. An *equilibrium of the abstract game* Γ is a strategy profile $s \in S$ which is jointly feasible (a fixed point of $F = \times_{j \in I} F_j$, i.e., $s \in F(s)$) and does not permit a feasible good reply, that is, $U_i(s) \cap F_i(s) = \varnothing$ for all $i \in I$.

<u>Theorem</u>. Suppose that for each *i*,

(1) S_i is a non-empty, compact, and convex subset of a Euclidean space,

(2) F_i is non-empty, continuous, compact, and convex,

(3) U_i has an open graph in $S \times S_i$, and

(4) $s_i \notin Hull[U_i(s)]$ for all $s \in S$

then the abstract game Γ has an equilibrium (Shafer & Sonnenschein, 1975)).

One may retro-fit elements of each type of game reviewed into the framework of an abstract game. Specializations of each game are abstract games with assumptions placed on each of the components, *I, S, P*, and *F*. quantum fuzziness, that is, the superposition of quantum states using membership functions, instead of classical probabilities, can be viewed as a generalization to quantum probability. However, a more accurate view of this would be the quantum structure of fuzzy sets (Mesiar, 1995). By the nature of fuzzy sets, fuzziness of quantum structures would entail point-wise operations on a global structure. In this regard, one would quantize fuzzy games. We then take the fuzzy game strategies and apply the feasible quantum operators in order to define final payoffs in terms of fuzzy values. Next, one would then defuzzify these values to crisp values in order to return to the realization of classical payoff values. If the macro-structure of all survival games is evolutionary, then by setting up a game as a very general evolutionary structure that consists of deterministic feedback and non-feedback (differential) and stochastic elements, and then apply quantum fuzzy operators, as micro-strategies, then a holistic structure begins to emerge for a general abstract game. Using the framework of an abstract game with the macro elements of differential and stochastic systems and the micro operators of quantum fuzzy systems, a connective meso-system can be produced.

Behavioral game theory and economic choice utilize behavioral and psychological aspects of human decision making in its strategy and payoff descriptions of a game (Camerer, 2003). Neuroeconomics blends the ideas of neuroscience and aspects of economic and behavior game theory, to endeavor to explain the way organisms make decisions in a biological landscape. Bayesian probability and relative preference ordering play large roles in the theories of this approach. For example, prior distributions that bias the way that some choices are presented enter into the equations for preference of strategies, along with preference ordering from payoff comparisons. In these approaches, order preferences rely on relative ranking based on a myriad of physical and psychological profiles of the value of goods. Research in this field has recently shown that the orbitofrontal cortex (OFC) in primate brains independently assign value to choice (Padoa-Schioppa & Assad, 2006). This functionality is separated from the choice bias that may be introduced by sensory or motor process input to other parts of the primate brain in forming a choice stratagem. Value assigned by humans may be quite abstract, as in "pure altruism" or the "warm glow" feeling that one obtains when one gives to charity with advance notice of not receiving any apparent physical or psychological reward. A recent result using fMRI on brains of college students subjected to games of taxation and shared charity showed that the caudate, nucleus accumbens and insula components in the brain displayed activity precisely when they were confronted with making such purely altruistic choices (Harbaugh, Mayr, & Burghart, 2007). The structure of the constraint and preference functionals, P and F, may be able to contain or better describe these phenomena in an axiomatic fashion. For example, P may have components that represent the various choice sieves in the brain, i.e., each component of the brain that affects choice stratagem based on a type of stimuli. F may then act as the inhibitor or excitor functions for these components. Both of these functionals can then be the basis for a new neurological substrate that can replace the familiar utility theory used in classical game theory. In addition, these functions can be quantum, fuzzy, and chaotic in structure, since they entail biological information.

This holistic approach to a super-structure for games and decision-processing is to be investigated in detail next. In the context of an abstract game Γ, we can define the preference structure according to a belief system of the agents, which may be behavioral (stochastic) and deterministic. Belief also entails fuzziness. The quantum nature of belief is manifested by a global quantum entanglement phenomenon. What this means is that the information an agent receives about all other agents through the play of strategies can be shared before realization.

Details of quantum fuzzy games must be researched in order to produce a better idea of how equilibriums can be computed. Under uncertainty, these models may be chaotic and as such, emergent behaviors need to be investigated as a whole. The idea that a general game involving organic and inorganic agents, each making holistic decisions about micro and macro strategies and connected by meso-strategies may define a more realistic organism. The interplay is not just the Butterfly Effect. It is the global game. Consciousness may be modeled in the large, by a holistic game structure.

Continuum Games

Consider the case where the roles of agents in a game are so small because of the shear size of the population of participants. Large markets, job industries, multi-cellular organisms of extreme complexity and size are some examples of these games in the limit. In such games, the number of agents will occupy a continuum set that is isomorphic to the unit interval $[0,1]$.

Def. A *continuum* or *non-atomic game* will consist of a $\sigma-$algebra, $\mathfrak{S}$ of subsets of $[0,1]$ and a real-valued function *v* defined on $\mathfrak{S}$, satisfying the following:

(i) $\nu(\varnothing) = 0,$

(ii) $\nu(A \bigcup B) \geq \nu(A) + \nu(B),\ A, B \in \mathfrak{S}$ and $A \bigcap B = \varnothing$.

The elements in $[0,1]$ are the agents and those of $\mathfrak{S}$ are the coalitions of the game. Typically, we shall have continuum games of the form $(A, \mathcal{A}, \mu)$, where $A = [0,1]$, $\mathcal{A}$ is the Borel $\sigma-$algebra, and μ is the Lebesque measure.

Def. The continuum game $(A, \mathcal{A}, \nu)$ is in (0,1) normalization if:

(i) $\nu(0,1) = 1$

(ii) $\nu(S) \geq 0 \qquad \forall S \in \mathcal{A}$

(iii) if $\alpha(S) = \inf \sum_{\substack{S_i \in \mathcal{A} \\ S \subset \bigcup_i S_i}} \nu(S_i)$, then $\alpha = 0$.

Def. An imputation for the game $(A, \mathcal{A}, \nu)$ is any (signed) measure σ satisfying:

(i) $\sigma([0,1]) = \nu([0,1])$

(ii) $\sigma(S) \geq \alpha(S) \quad \forall S \in \mathcal{A}$

Def. The measure σ *dominates* τ, written $\sigma \succ \tau$, if $\sigma(A) > \tau(A)$ for all $A \in \mathcal{A}$ such that $\sup_{B \in \mathcal{A}} \{\nu(A \cup B - \nu(B)\} > 0$.

Definition. For any game *v*, a *null set* is $[0,1] \setminus S$ for some carrier *S* of *v*. An *atom* is any measurable non-null set *S* such that, if $S = T \cup Q,$ where $T \cap Q = \varnothing$, with *T* and *Q* being measurable sets, then either *T* or *Q* is a null set.

Def. A game *v* is *non-atomic* if it contains no atoms.

The following is a framework and requirements synonymous with the Shapley value for a continuum game *v*:

Given a continuum game v, the value of *v* is a finitely additive set function (signed measure) $\varphi[v] : \mathcal{A} \to [0,1]$. If θ is an automorphism of $[0,1]$, then, for any *v*, we define the game $\theta_* v$ by $\theta_* \nu(S) = \nu(\theta(S))\ \forall S \in \mathcal{A}$. φ satisfies the following (modified Shapley axioms):

(i) For any v, $\varphi[\nu]([0,1]) = \nu([0,1])$

(ii) For any game *v* and automorphism θ, $\varphi[\theta_* v] = \theta_* \varphi[v]$.

(iii) For games *v*, *w*, and scalars α, β, $\varphi[\alpha v + \beta w] = \alpha\varphi[v] + \beta\varphi[w]$.

(iv) If *v* is monotone, then $\varphi[v]$ is also monotone (Owen, 1995, p.348).

An existence and uniqueness statement about φ defined on the space of all non-atomic games cannot be said (there are counterexamples and the space is too large to accommodate closedness under any norm). However, by selecting a suitably large and useful subspace of the space of non-atomic games, such an existence and uniqueness theorem can be shown. One can then proceed with Shapley-like analysis, as in the finite agent population cases. To this end, we define the following:

Def. Let $S^{(m)} = \{S_0, S_1, ..., S_m\}$ be a sequence of measurable sets in $\mathcal{A}$, satisfying $\varnothing = S_0 \subset S_1 \subset ... \subset S_m = [0,1]$. $S^{(m)}$ is called a *chain* of sets. Define a number using the space of such finite chains in $\mathcal{A}$ as $\|v\|_{S^{(m)}} = \sum_{i=1}^{m} |v(S_i) - v(S_{i-1})|$. Letting $m \to \infty$, we approach infinite chains in $\mathcal{A}$ and appropriately define a new number as $\|v\| = \sup_{S^{(m)} \in \mathcal{A}} \|v\|_{S^{(m)}}$. $\|v\|$ is the total variation of the function *v*.

Def. Denote the space of all functions with finite total variation on [0,1], i.e., bounded variation on [0,1] by

$$BV[0,1] = \{ \nu \ | \ \|\nu\| < \infty \} .$$

Theorem. $(BV[0,1], \|\nu\|)$ is a Banach space.

Def. Define a closed subset of $BV[0,1]$, denoted by $bvNA[0,1]$, as follows; define the following spaces:

$$NA^{+}[0,1] = \{ \mu \mid \mu \text{ non-atomic measure on } [0,1], \mu([0,1])=1 \}$$

$$cBV[0,1] = \{ f \mid f \in BV[0,1], \text{ continuous at 0 and 1 with } f(0) = 0 \}$$

then

$$bvNA[0,1] = \{ v \mid v = f \circ \mu, \ f \in cBV[0,1], \mu \in \mathrm{NA}^{+}[0,1] \}$$

Theorem. There exists a unique value, φ, on the space $bvNA[0,1]$ satisfying the modified Shapley axioms for a continuum game.

Fuzzy sets defined on $[0,1]$ can be described as the set of Borel-measurable functions $f:[0,1] \to [0,1]$ (into mapping). Now define the set of all such fuzzy sets as F. A fuzzy set function is then a mapping $\nu^{*} : F \to \mathcal{R}$. It can be shown that $\nu^{*}(\chi_S) = \nu(S)$, where χ_S is the characteristic function of the set S and $\nu \in pNA$, where

$$pNA = \{ \nu \mid \nu = f \circ (\mu_1, ..., \mu_n), \mu_i \in NA^{+}, \ f \in C^{2}([0,1]^{n}), \ f(0) = 0 \}$$

Furthermore, the existence and uniqueness of fuzzy set functions v^* defined on *pNA* that satisfy modified Shapley axioms:

(i) $(\alpha v + \beta w)^* = \alpha v^* + \beta w^*$,

(ii) $(uv)^* = u^* v^*$,

(iii) $\mu^*(f) = \int\limits_{[0,1]} f d\mu$

(iv) v monotonic $\Rightarrow$ v^* monotonic

where $v, w \in pNA$, $\alpha, \beta \in \mathcal{R}$, $\mu \in NA$, and $f \in F$ can be shown.

Behavioral Game Theory

Nash equilibrium and other tenets of classical game theory assumed the rationality of the agents of a strategic game. Where organisms with reflection are concerned, rationality may escape the situation rapidly. Humans inevitably employ emotion, mistakes, limited or incomplete foresight, unknown or imprecise opinions of others and their respective abilities, and adaptive learning. It has been said that Analytical Game Theory is to the social sciences what the periodic table is to physical chemistry, taxonomy of strategies for societies (Camerer, 2003). Behavioral Game Theory endeavors to empirically approach real world non-cooperative games and contrast their unfolding sage with the predictions from Game Theory. In this regard, and as a whole, it has been observed that Game Theory is half right and half wrong. What has been observed is that Mixed Strategy Equilibriums (MSEs) discussed before as probabilistic mixtures of pure strategies applied as the agents strategy space, is asymptotically accurate for many real world gaming. Strategies seem to fluctuate around this rule. A popular alternative to MSE has been the usage of the quantal response equilibrium (QRE). In this strategy an agent does not choose the best response with probability 1. Rather, they choose a better responder, not looking for the ultimate prize payoff each time, as in Nash equilibrium. Analytically and typically, a QRE uses payoff response functions of the form (logit or exponential):

$$p(s_i) = e^{(\lambda \sum_{s_{-i}} p(s_{-i}) u_i(s_i, s_{-i})) / \sum_{s_k} e^{\lambda \sum p(s_{-i}) u_i(sk, s_{-i})})}$$

Using this, agents fix a strategy and form heuristics about other agents, computing expected payoffs given those heuristics. The parameter λ is used as a noisy measure of sensitivity to differences in expected payoffs. $\lambda = 0$ means strategies are chosen equiprobably. Nash equilibrium strategies are akin to using infinitely large λ . QRE is then an intermediary between randomization of strategies and Nash Equilibrium. In other words, QRE can be used to roughly approximate irrational strategy development given a noisy environment of knowledge and emotion. Generally, behavioral games can be categorized under one or a combination of the following scenarios:

(1) Simple bargaining games: ultimatum, dictator, or trust games

(2) Mixed strategy games

(3) Bargaining games

(4) Dominance-solvable games

(5) Learning games

(6) Multi-equilibrium coordination games

(7) Signaling and reputation games

Under simple bargaining games, Dictator games (single decisions) have an agent dictate the division of a pot between themselves and other agents, the so-called “take-it-or-leave-it”. Ultimatum games are dictator games that give the responder agents a chance to reject the offer from the proposer. If the responder rejects the proposer’s offer, the payoff is zero for all. In trust games, the amount to be distributed by the dictator (in this case a trustee) to all is determined by an amount invested by a third party, the investor. Prisoner’s dilemma (PD) and public goods (PG) games are well examined examples of these types of games, but they may not yield analytical information about the agents’ social choices. Experiments have shown that 10-20 percent of agents show a small amount of altruism in dictator games (they offer more than would otherwise be expected from a dictator’s point of view). Alas, they also act out negative reciprocity, rejecting offers in ultimatum games that are less than 20 percent with a probability of 0.5. Because of this, proposers will offer 30-50 percent in ultimatum games. Investors in trust games risk nearly half their original investment and in turn usually receive nothing, i.e., make no profit. On the whole positive reciprocity is weak compared to negative reciprocity. That is, people are quicker to avenge attacks than to thank someone (Camerer, 2003). In “inequality-aversion” models, agents are concerned about their own payoffs and in the differential between their payoffs and those of others (Fehr & Schmidt, 1999). If

$x = (x_1, ..., x_n)$ is the allocation vector amount for an *n*-agent game, then the “inequality aversion” model describes agent *i*’s utility as:

$$U_i(X) = x_i - \frac{\alpha}{n-1}\sum_{k \neq i}\max(x_k - x_i, 0) - \frac{\beta}{n-1}\sum_{k \neq i}\max(x_i - x_k, 0) \qquad (3.7)$$

where $0 \le \beta_i < 1, \beta_i$ is a “guilty” weight for having a higher allocation than others, and $\beta_i \le \alpha_i$ where α_i is an “envy” weight for disliking having a lower allocation than others. Another measure of inequality aversion is the “equity, reciprocity, and competition” model (ERC) of Bolton and Ockenfels (2000):

$$U_i(X) = U\left(x_i, \frac{x_i}{\sum_{k=1}^{n} x_k}\right) \qquad (3.8)$$

In this model, agents strictly prefer a relative payoff that is equal to the average payoff $\frac{1}{n}$. In other words, in an ERC model, agents endeavor to not show up on the “radar screen” for outlandish rewards or losses. This may also be a form of simultaneous risk aversion and egalitarianism (Kroll, 2006). A third form of inequality aversion is Rabin’s fairness equilibrium given as follows: in a 2-agent game, let b_{3-i} denote agent *i*’s belief about the other agent. Let $\pi_i^{\max}(b_{3-i})$, $\pi_i^{\min}(b_{3-i})$, and $\pi_i^{fair}(b_{3-i})$ be the maximum, minimum, and fair payoffs respectively for agent *i* from the other agent based on the preconceived beliefs. Then agent *i*’s “reciprocal kindness” toward the other agent, given an action of a_{3-i} by that other agent is:

$$\kappa_{3-i}(a_{3-i}, b_i) = \frac{\pi_i(b_i, a_{3-i}) - \pi_i^{fair}(b_i)}{\pi_i^{\max}(b_i) - \pi_i^{\min}(b_i)} \tag{3.9}$$

Agent *i*'s perceived kindness of the other agent toward them is:

$$\lambda_i(b_{3-i}, c_i) = \frac{\pi_i(c_i, b_{3-i}) - \pi_i^{fair}(c_i)}{\pi_i^{\max}(c_i) - \pi_i^{\min}(c_i)}$$

Then the social preference of utility for agent *i* is given as:

$$U_i(a_i, b_j, c_i) = \pi_i(a_i, b_j) + \alpha\lambda_{3-i}(b_{3-i}, c_i)[1 + \kappa_i(a_i, b_{3-i})]$$

where α weighs fairness against money. The "fairness equilibrium" is the case where $a_i = b_j = c_i$, i.e., when beliefs of others are correct, beliefs about what others believe are correct and all agents maximize social utility (Camerer, 2003, 106). An extensive form extension to Rabin's "fairness equilibrium" is given by Dufwenberg and Kirchsteiger (2004). In this extension, kindness functions are defined as differences between payoffs and fair payoffs. Here we will generalize the notation to *N*-person finite games. The measure of agent *i*'s kindness to agent *i* is then given by:

$$\kappa_{ij}(a_i(h), (b_{ij}(h))_{j \neq i}) = \pi_j(a_i(h), b_{ij}(h))_{j \neq i}) - \pi_j^{fair}((b_{ij}(h))_{j \neq i})$$

$$\lambda_{iji}(b_{ij}(h), (c_{ijk}(h))_{k \neq j}) = \pi_j(b_{ij}(h), (c_{ijk}(h))_{k \neq j}) - \pi_j^{fair}((c_{ijk}(h))_{k \neq j})$$

where

$$\pi_j^{fair}((b_{ij})_{j \neq i}) = \frac{\pi_j^{\max}(a_i, (b_{ij}(h)_{j \neq i}) - \pi_i^{\min}(a_i, (b_{ij})_{j \neq i})}{2}$$

The utility function is then given by:

$$U_i(a_i(h),(b_{ij}(h),(c_{ijk}(h)_{k\neq j})_{j\neq i}) = \pi_i(a_i(h),(b_{ij}(h))_{j\neq i}) + \sum_{j\neq N\backslash\{i\}}(Y_{ij}\lambda_{iji}(b_{ij}(h),(c_{ijk}(h))_{k\neq j}))\kappa_{ij}(a_i(h),(b_{ij}(h))_{j\neq i})$$

where *h* denotes the stage of history of the game, i.e., were the agents are in the decision branch of the game, $b_{ij}(h)$ is the belief that agent *i* has about agent *j*'s kindness to agent *i*, and Y_{ij} is an exogenously given non-negative number for each pair *(i,j)* that measures how sensitive agent *i* is to the reciprocity concerns regarding agent *j*, i.e., how sensitive one agent is towards the feeling about kindness/unkindness towards another agent (Dufwenberg & Kirchsteiger, 2004). Note the differences between this development and that of Rabin's reciprocity. Scaling is absent; instead a sum across all agent of the product of reciprocal kindnesses is done. The kindness of another agent, *j*, does not figure directly in the calculation of utility. Dufwenberg and Kirchsteiger proceed to define a different kind of equilibrium that they label as sequential reciprocity equilibrium (SRE) in which agents optimize social utility, U_i , and in addition, strategies match the belief vector functions, b_{ij}. The existence of an SRE is proven under finite *N*-person games. From our perspective, the exogenous variable Y_{ij} can be distributed according to a pdf or Bayesian belief system. All beliefs are conditional, as probabilities are, if given from a Bayesian viewpoint. However, these belief frameworks are adaptive in humans. Experiential processes constantly shift our attitudes toward externalities. These shifts are learning algorithms and as such, human games are multi-stage in nature, unless the terminal nodes of one game lead to extinction or death, the so-called stopping games mentioned before. Therefore, the games demonstrated by Behavior Game Theory introduce the psychological and bounded rational elements of human decision-making into multi-stage stopping games. In another model of reciprocity, Falk and Fischbacher (2006) use both outcomes and intentions about beliefs of kindness in calculating strategy equilibria, defining emotional terms at each node, *n*, of the game. Kindness (fairness measure) is denoted by:

$$\Delta(n) = \pi_i(n, 2(s_i), 2(s_j)) - \pi_j(n, 2(s_i), 1(s_j))$$

where s_i is *i*'s strategy, π_i is *i*'s payoff, $1(s_i)$ is *i*'s belief about *j*'s choice (first-order belief), and $2(s_i)$ is *i*'s belief about *j*'s belief about *i*'s choice (second-order belief). Kindness is then the difference between an agent's expected payoff and that of others, based on first and second-order beliefs. Next, an *intention function,* $\Omega(\pi_i, \pi_j, \pi_i^0, \pi_j^0)$ is given that compares a pair of payoff functions, (π_i^0, π_j^0) with an alternative pair, (π_i, π_j) in the following manner:

$\Omega(\pi_i, \pi_j, \pi_i^0, \pi_j^0) = 1$ if (1) *j* gives more to *i* than to themselves when, in fact, *j* could have given *i* less, or (2) when *j* gives less to *i* than to themselves when, in fact, *j* could have given *i* more, and $\Omega(\pi_i, \pi_j, \pi_i^0, \pi_j^0) = \varepsilon_i$ for some value ε_i, if (1) *j* gives more to *i* than to themselves when, in fact, *j* could have given *i* even more, or (2) when *j* gives less to *i* than to themselves when, in fact, *j* could have given *i* even less. Finally, an intention factor, $\upsilon(n, 1(s_i), 2(s_i))$, that depends on the node *n*, and the first and second-order beliefs, $(1(s_i), 2(s_i))$ and given the payoff pair, (π_i^0, π_j^0), is defined as:

$$\upsilon(n,1(s_i),2(s_i)) = \max_{(\pi_i,\pi_j)} \Omega(\pi_i, \pi_j, \pi_i^0, \pi_j^0)$$

where the maximum is taken over all possible alternative pairs, (π_i, π_j) available at node *n*. Now suppose that there is a path from node *n* to an end or terminal node *t* in a finite game branching. Call the next node in this path *M(n,t)*. Define *i*'s reciprocation of *j* by:

$$\sigma(n,t) = \pi_j(M(n,t),1(s_i),2(s_i)) - \pi_j(M(n,t),1(s_i),2(s_i))$$

Agent *i*'s utility at a terminal node *t* is then given by:

$$U_i(t) = \pi_i(t) - \rho_j \lim_{n\to t} \sum_n \upsilon(n)\Delta(n)\sigma(n,t)$$

The limit is taken over all nodes *n* that precede the terminal node *t*. In this measure, the physical payoff is reduced by the *human emotional payoff* that takes into account measures of kindness, intentionality, and reciprocation along all nodes on a path to a terminal node *t*. Here, the value, ρ_i is a weight that *i* assigns to the emotional payoffs (Falk & Fischbacher, 2006). Again, we mention that a more appropriate weight may be stochastic with respect to a Bayesian pdf. The values ε_i, represent subjective measures of intention, that is, what agents think relatively about how others could have treated them better or worse on other paths or occasions, hence the intentionality.

We switch to mix strategy games again, visiting the notions of patent race and location games. Patent races are investment games in which a group of firms may invest any portion of a fixed endowment, *e*, towards the development or sponsorship of an idea or product. If *B* represents the budget of an organization, then after investing *e* in a product, $B - e$ is left for the firm. The dynamics of game are as follows: the firm that invests the most amount of endowment is awarded a fixed prize, *r*; if the firms spend the same amount, no firm earns a prize. There exists a unique symmetric MSE where firms spend the total endowment, *e*, given to them for the product with probability $\frac{r-e}{r}$, and smaller amounts with uniform probability $\frac{1}{r}$, including the choice of no investment, $e = 0$. Empirical results show that the MSE is surprisingly accurate in predicting the game choices of participants. While participants usually picked $e = 0$ too often, the most common case was to invest all of the endowment (Rapoport & Amaldoss, 2000).

Bargaining games include those games in which economic agents agree on pre-defined terms of agreement usually for trading. Structured and unstructured bargaining games are possible. In unstructured games, agents decide on the type of messages to be sent to other agents, the order of the offers, and other facets of bargaining. In structured games, an experimenter decides on the bargaining details. More human behavior occurs with unstructured bargaining games. Experiments have shown that agents gravitate toward focal divisions in unstructured bargaining. In the case of exchange of tokens for money utility, focal points are directly competing. Having two focal points converges the game dynamics into a bimodal distribution in agreements and a general increase in the disagreement rates in the negotiating. Self-serving biases are self-fulfilling prophecies, that is, biased bargainers will usually collect or listen more closely to their perceived more important information, neglecting a more diverse spectrum and distribution of input. The isolated bargainer, in this case, the biased bargainer creates counter-productive coalitions. The common thread in studies of unstructured bargaining games in reality is that offers and counteroffers gravitate between an equal split of the utility and those predicted by classical game theory. In addition, agents do not use induction as a tool for learning about negotiations, as much as theory would predict. We do not look back too much when endeavoring in negotiations.

We revisit dominance-solvable games in a more general context. In dominance-solvable games, agents assume that others will act rationally in measuring what is a dominant strategy and will act accordingly. Knowing this to be true, one can then make an educated guess about what others will do. They will essentially all respect the dominance credo of strategies. If this be the case, then by iterating through each strategy possibility, an agent can move toward eliminating dominated strategies for other agents, opening up possibilities for initially un-dominated strategies to become dominated. If this sequence of dominance iterating leads to an equilibrium then the game is called *dominance-solvable*. Dominance solvable games contain the familiar iterative guessing game of "she thinks that I think that she thinks that I think that, ad infinum). This can be modeled by the concept of *n*-level agents: an *n*-level agent thinks that all others are *(n-1)* level agents who, in turn, think that all others are *(n-2)*-level agents, ad infinum. Studies show that agents utilize two to three steps of iterations, employing dominance measurability for their own decisions, but not for others after these iterations (Camerer, 2003). Learning may be key to whether these games lead to successful outcomes. Several methods depend on experience-weighted attraction (EWA), the method of weighing the experience of choosing various strategies and what was actually paid from them.

Attraction here means a numerical evaluation assigned to a strategy after the game play. On the spectrum of experience are the extremes of belief learning in which no particular attention is paid to what was actually learned and reinforcement learning in which only the experience of payoff history and corresponding strategies is used. More specifically, let $A^S_{(t-1)}$ denote the attraction of strategy *S* before play at time *t-1*. Reinforcement updates attractions by:

$$A^S_t = \phi A^S_{(t-1)}(t-1) + (1-\varepsilon_t) - \rho(t-1)$$
$$A^{S+\delta}_{t-1} = \phi A^{S+\delta}_{(t-1)}(t-1) + \varepsilon_t$$

where the strategies $S+\delta$,are neighboring or similar strategies to *S*, ε_t represents a generalization of reinforcement from *S* to $S+\delta$ at time *t*, and ρ is a damping factor for the prior time from the present. These are cumulative effects. One can also use weighted attractions that are not cumulative, but are dependent on belief weights alone, i.e.,

$$A^S_t = \phi A^S_{(t-1)}(t-1) + (1-\phi)(1-\varepsilon_t)$$
$$A^{S+\delta}_{t-1} = \phi A^{S+\delta}_{(t-1)}(t-1) + (1-\phi)\varepsilon_t$$

Models of reinforcement can generalize ε_t based on actual payoff differentials, lagged attractions and learning rates. Learning Direction theories endeavor to predict the direction of changes in strategy choices. They cannot point to specific strategies, just the direction of change and hence can be of limited use since the topology of strategies can be such that neighboring strategies that are in the direction of improvement can still be very bad choices. Imitation is another method of strategy learning. In imitation, as one might guess, successful strategy regimes are copied by others in the hopes of following the footsteps of a perceived successful strategist. Imitation more often, does not require that any long-term beliefs be formed, instead relying on only successes. Local evolutional models use imitation, as well as animals and children. Imitation is also used as a rule when agents are under severe time constraints to make decisions.

Hybrid methods in which weighted versions of both types of learning seem to better predict game scenarios. Rule learning is a technique in which learning rules (not strategies, but rules for choosing strategies) are chronicled closely and strategies gradually shift toward those that exhibit superior performance (Stahl, 2000). One defines a behavioral rule as a map $r:\Omega^t \to \Delta(A)$ where Ω^t is the information available at time *t* in the game and $\Delta(A)$ is the set of probability measures defined on the action space *A*. Define an array of evidence or scores: $y^R = (y_0,,,y_n)$ where y_i is the "evidence score such as a weighted average of previous play by others a form of imitation, belief learning, reinforcement, etc, and the last evidence, y_n, is the Nash payoffs. Since the space of rules may be continuous, we can generalize these to be functions parameterized by θ, mapping the information available during a game to an evidence score, $Y(\Omega^t,\theta,)$. Assign weights $\alpha = (\alpha_i)_i$ that reinforce evidence y_i. A weighted-evidence array is then $\alpha\left(y^R\right)^T \in \alpha Y(\Omega^t,\theta)$. One can then use a probability distribution that assigns a probability of choosing an action *j* based on these evidence-weights. Stahl using a logit probability distribution.

Games in which multiple equilibrium exists necessitate a coordinated effort among the agents to a selection criteria. Three types of coordination are prominent: (1) pure matching games, in which all equilibria have the same payoffs for an agent, in which case other subjective methods are used, (2) assurance games in which the payoff-dominant equilibrium are usually risky, and (3) battle-of-the-sexes (BOS) games, in which equilibria are chosen based on non-technical or objective personal preferences. In studies, agents converge toward payoff-dominant equilibria regardless of the type of games involving coordination. In BOS games, if a leader emerges among the agents, announcing a common coordination schema to others, failure is less likely. Pre-play (before a game starts) coordination also lessens the possibility of failure. If more than one agent contend for this type of leadership, arguments ensue and coordination usually fails.

In signaling games with asymmetric information (some agents know more about the game than others), the sender may send private information to any receiver agent. Studies show that agents converge invariably to signal equilibria, but the details of which equilibrium often remain obscure and less likely equilibria emerge as those reached. Order-statistic games are versions of signaling because in such games a number is chosen by an agent and their payoff depends on a statistic computed from those numbers picked, i.e., the medium, mode, minimum, maximum, etc. Here the payoff for agent *i* is defined as $\pi_i(s_i^j, s_{-i}^k) = f(s_i^j, F((s_{-i}^j)_j)]$, where F is some statistic of the vector of strategies, $(s_k^j)_{kj}$, and f is some well-defined function of the agent strategy s_i^j and of the statistical estimate $F((s_{-i}^j)_j)$. The signals send are the actual numbers chosen by other agents.

Knowing that agents are picking numbers tells the agent something about the statistic and hence of the payoff possible.

Game theory is inherently a story about the survival of entities using decision strategies, against the backdrop of an anti-universe of competing decision landscapes. The universe has been more succinctly described by emergent properties of many-particle quantum systems, uncertainty within the fuzzy or probabilistic structures, the evolutional rules of engagement, and macro stochastic and differential feedback systems. Chaos, as a separate emergent property has not been studied in the context of a macro-game. A holistic game should incorporate chaos, as a possible manifestation of value. This new game structure embodies the holographic approach as well. Seen from a large-scale lens, games emerge as evolutionary processes, with inherent chaotic behavior. Microscopically, though, strategy decisions may be quantum fuzzifications of classical decisions. In order to consolidate all physical scales, in the tradition of theories of everything (TOEs) in physics, an analogy in game structures should also simultaneously handle quantum, relativistic, and nonclassical logics environments for games. This is considered in quantum-gravity-induced games as will be discussed in the main section in terms of causaloid-based QG and a general framework for uncertainty from Zadeh (2006).

Generalized Game Theory and Social Games

Burns and Roszkowska (2001; 2005) propose a generalized approach to game theory that more completely accommodates psycho-social and cognitive-judgmental structures. Their generalized game theory (GGT) entails social structures as a subcomponent of a game. Rule complexes generalize game state transitions by meta-modeling more recent cognitive rule structures. The approach of this study is to utilize Zadeh's GTU constraints, a higher order logic representation of uncertainty to explicitly express general uncertainties.

Rule complexes *C*, consist of a set of rules and/or rule complexes. Rule complexes are generalizations to rule sets. Rule complexes are closed under set-theoretic union, intersection, differences, and power set generation, and they preserve set inclusion, (i.e., if $C_1 \subseteq C_2$ and $C_2 \in C$ then $C_1 \in C$, and if $C_1, C_2 \in C$ then, $C_1 \bigcup C_{2,} C_1 \bigcap C_2, C_1 \setminus C_2, P(C_1) = 2^{C_1} \in C$). If *B* is a complex, a subcomplex *A*, $A \subseteq B$ can be generated from *B* by deleting rules and/or redundancy from *B*. Rule complexes can express interdependencies of rules and hence generally, social relationships and organizational connectives. Essentially, rule complexes generalize state transition rules in an attempt to model complex social network relationships. GGT games then use the class *C* to replace classical state transitions. Situational context is also taken into consideration in time. In this way, a GGT game *G(t)* is time situational dependent. Agent's social roles are embodied by a component of the rule complex.

Briefly, we outline the components of a situational GGT game, *G(t)*. Let $ROLE(i,t,G)$ denote agent *i*'s role expressed as a rule complex at time *t*. A situational game is then expressed as $G(t) = \left\{ \left\{ ROLE(i,t,G) \right\}_{i \in I}, R \right\}$ where *R* is a general rule complex that handles payoff rules (among other non-agent components of *G(t)*) and *I* is the index of agents. Involved in the general role complex is a subcomplex of agent belief models or frames given by $MODEL(i,t)$. Agent values are expressed as the subcomplex $VALUE(i,t)$. Agent strategies are given by a subcomplex $ACT(i,t)$ and judgment subcomplexes are expressed as $J(i,t)$. In behavioral game strategies, $J(i,t)$ houses the complex of rules about how agents arrive at truth values, validity, value and eventually, the choice of strategies in given time situations. One then has the rule complex inclusion chain,

$$\begin{aligned}\{MODEL(i,t), VALUE(i,t) ACT(i,t), J(i,t)\} &\subseteq_g ROLE(i,t)\\ &\subseteq_g ROLE(I,t)\\ &\subseteq_g G(t)\end{aligned}$$

where $\subseteq_g$ is the set inclusion for rule complexes in a GGT game.

When actualizing actions (which compose strategy profiles), one considers a value, v, from $VALUE(i,t)$, that aligns with the selection of an action a. The essential qualities or qualia of v is denoted by $Q(v)$. The expected qualia of taking action a, is denoted by $Q_E(a)$. A judgment rule from $J(i,t)$ is applied to a measure of the similarity of $Q_E(a)$ to $Q(v)$. Let B denote the actions which are candidates for actionization by the agent. Denote a similarity divergence measure D_Q, which differentiates qualia between value goals and actions. The action b^*, to take in B is that which maximizes the judgment operator resultant,

$$\mathrm{b}^* = \arg\max_{b \in B} J(i,t)\left[D_Q\left(Q_E(b), Q(v)\right)\right] \tag{4.48}$$

The action b^*, is a *satisfier* of v in the tradition of Simon (1957). D_Q defines a preference ordering for value/action (dis)similarities. For an agent i, $B = ACT(i,t)$.

In asymmetrical two-coalition games, agents prefer to maximize the differences between outcomes or payoffs between themselves and the other coalition provided agent stability persists inside each coalition. Their judgment rules would then weigh actions from the best response space of actions, $ACT(1,t), ACT(2,t)$ for situations at time t such that,

$$\left(a_1^*, a_2^*\right) = \arg \max_{\substack{a_1 \in ACT(1,t) \\ a_2 \in ACT(2,t)}} \left\{ J(1,t)\left[D_Q\left(Q_E\left(a_1\right), Q\left(v_1\right)\right)\right] + J(2,t)\left[D_Q\left(Q_E\left(a_2\right), Q\left(v_2\right)\right)\right]\right\} \tag{4.49}$$

While classical closed games are structurally static, (i.e., fixed agents, rules, etc.), the concept of an open game is introduced in GGT to accommodate changing social attitudes of groups of agents. In open GGT games, role complexes may evolutionarily change based on diverse social constructs. In inception games, the coalitions may evolve to different (non)cooperative rules of engagement within and outside of each coalition. The concept of *n*-agency may dramatically change the dynamics of intra and inter-coalition behavior. Game solutions take on a different meaning in terms of satisficing. Common solutions in GGT are then strategy profiles that result in the satisficing of all agents. An analogous version of Nash equilibrium for GGT follows, Def. Let G be a social game in GGT, I an index of agents, and $S = \{s_i\}_{i \in I}$, the set of corresponding strategies. A strategy profile $a_I^* = \left(a_i^*\right)_{i \in I}$ is a *Nash equilibrium* in pure strategies in G if,

$$J(i,t)\left[D_Q\left(Q_E\left(a_i^*\right), Q\left(v_i\right)\right)\right] \geq J(i,t)\left[D_Q\left(Q_E\left(b, -a_i^*\right), Q\left(v_i\right)\right)\right],$$

for all agents $i \in I$ and alternative strategies $b_i \in s_i$. Schilling focal points can then be chosen based on uniform satisficing among all agents in role complexes.

For this study, we consider the general expression of uncertainty from Zadeh (2006) applied to the various rule complexes that comprise $ROLE(\mathrm{I}, t)$ and R in G. Refer to Appendix C for details on the GTU and generalized constraints as metaexpressions of general uncertainty.

Linguistic Geometry and Large-scale Hypergames

Large scale games in which the number of interacting agents and strategies produce computational explosiveness when searching for equilibria present a practical problem in complex adaptive systems modeled by extended game forms. Inceptions, may present with very complex interaction scenarios. Even in the presence of manageable numbers of agents, the landscape of possible dynamic stratagem, payoffs, and uncertainty regimes in inceptions manifest computationally impractical equilibria searches. Stilman (2000; 2011) and Stilman, Yakhnis, & Umanskiy (2010) develop the concept of hypergames as linked abstract board games using a geometric space definition of knowledge representation and reasoning. Linguistic geometry (LG) is a game-theoretic approach to solving for solutions (equilibria) in large-scale extensive-form discrete games. LG was formed experimentally based on the strategies of chess players in human reasoning which is extended to apply to general tactical warfare and adversarial reasoning games. First one defines ABGs:

Def. *Abstract board games* (ABGs) are tuples $\Sigma = \left(X, P, R_p, \Gamma, v, S_0, S_t, \mathrm{T}\right)$ where, $X = \{x_i\}_{i \in I}$ is a finite set of relatively positioned points (cell locations) on an abstract board; $P = \{p_i\}_{i \in J}$ is a finite set of (agent's) pieces belonging to one of two coalitions defined, $P = P_1 \cup P_2$, with possible asymmetric teams, (i.e., $|P_1| = J_1 \neq J_2 = |P_2|, J = J_1 + J_2$); $R_p\left(x, y\right)$ is a set of binary relations of reachability in X $\left(x, y \in X\right)$ using the pieces *P*, (i.e., R_p indicates whether a position *y* is reachable from another position *x* by the piece [agent] *p*); Γ is the state space; *v* is a non-negative function representing the values of the pieces; S_0 and S_t are the sets of start and target states respectively where $S_t = S_t^1 \cup S_t^2 \cup S_t^3$, the $S_t^i, i = 1, 2, 3$ are mutually disjoint, and $S_t^i, i = 1, 2$ are the target state spaces for $P_i, i = 1, 2$ respectively, with S_t^3 being the target state space of draw states (opposing sides have equal payoffs); and finally, T is a set of operators, $\rho_{(x,y,p)}\left(\alpha_t\right) \to \alpha_{t*}$ acting on the state space Γ with current state α_t, dependent on the space $X^2 \times P$, (i.e., if *p* moves from position *x* to *y*), producing another state α_{t*}.

The space T therefore contains all state transition operators of the ABG Σ, describing transitions from one state to another when a piece is moved from one location to another. Here, we differentiate using a time dependent definition of placement. Operators in T are composed (using conjunction) utilizing the relation operators R_p and time placement operators $ON_t : P \to X \bigcup O$ defined as $ON_t(p) = x \Leftrightarrow$ piece *p* is located in *x* at time (stage) *t*. The location space *O* is defined as the offboard. Pieces from different agents cannot simultaneously occupy the same location (the relinquished piece is placed in *O*) and pieces cannot be placed in a location that is not reachable from its previous occupancy location. The description of the state of Σ at time or stage t, α_t, is given by evaluating the $|P|$-length sequence $\{ON_t(p)\}_{p\in P}$ at *t*, (i.e., $\alpha_t = \{ON_t(p)\}_{p\in P}$). The space $P \times \alpha_t$ may be viewed as the time resource space for game agents at *t*.

It is obvious that an ABG's pieces are specialized finite state machines (FSMs). The goal of P_i is to reach a state $\kappa \in S_t^i \bigcup S_t^3$ respectively. The optimal strategy of Σ is one that has transitions starting from a start state $\kappa_0^i \in S_t^i$ reaching a target state $\kappa_t^i \in S_t^i$ for $i = 1, 2$, given that the other side (-*i*) makes stochastic moves. The composite state space Γ is a subspace of a Cartesian product involving three components of the ABG Σ, namely : (i) the product of the state spaces of the cells, given by each cell component state space Λ_{x_i} for the *i-th* cell, (ii) the product of state spaces of each piece given by Φ_p for the piece p, and (iii) the space $\Psi = \{ f_\alpha : X \to 2^P \mid \alpha \text{ state} \}$ of possible functions f_α

that map pieces *p* on *x*, while in state α (i.e., $f_\alpha(x) = \{ p \mid p \text{ is in cell location } x \text{ at state } \alpha \}$).

This super-space of possible composite states is expressed as:

$$\Omega = \prod_{i \in I} \Lambda_{x_i} \times \left\{ \left(f_\alpha, \prod_{p \in \bigcup_{i \in I} f_\alpha(x_i)} \Phi_p \right) \middle| \; f_\alpha \in \Psi \right\} \tag{4.50}$$

One thus has the possibility inclusion $\Gamma \subset \Omega$. The set of ABGs is partitioned into three types or classes: (i) *alternating serial* (AS) systems in which only one piece can be moved at a time and opposing sides alternate, (ii) *alternating concurrent* (AC) systems in which all, some or none of the pieces of one side can move simultaneously and opposing sides alternate moves with possible relinquished opposition at the destination of concurrent moves, and (iii) *totally concurrent* (TC) system in which all, some, or none of the pieces of both sides can move simultaneously or be relinquished.

Hypergames are defined as interlinked AC ABGs where pieces and locations are linked. This interlinking then creates a common point spread across linked regimes of games. Trajectories are paths for pieces navigating within these hypergame scenarios. These trajectories are represented as strings of symbols $(cx_0)(cx_1)...(cx_l)$ taken over the alphabet $\{c\} \times X$ where *c* is a symbol, *X* is the abstract board of an ABG and *l* is the length of the string. Additionally, $R_p(x_i, x_{i+1}) = 1$ for $i \in I$, (i.e., each cell is reachable from its previous sequential cell). Let $\mathrm{T}_p(x, y, l)$ denote the set of trajectories with the same tuple (p, x, y, l) and $P(s) = \{x_0, x_1, ..., x_l\}$ to be the set of parametric values of the trajectory (string) *s* labeled above. Let $L(s) = l$ denote the length of the trajectory *s*. We label the shortest trajectory in $\mathrm{T}_p(x,y,l)$ as $\breve{T}_p(x,y,l) = \underset{s \in T_p(x,y,l)}{\arg\inf}\{L(s)\}$. An *admissible trajectory of degree k* in $\mathrm{T}_p(x,y,l)$ is the trajectory in $\mathrm{T}_p(x,y,l)$ which can be divided into *k* substring shortest trajectories. Control grammars formed from these trajectories and labeled Δ_Σ, are then used to construct types of trajectory patterns based on rules of engagement and movement of pieces. We give the definition of a control grammar for completeness.

Def. A (language) *controlled grammar* is an 8-tuple $G = (V_T, V_N, V_{PR}, E, H, PARM, L, R)$ that extend other grammars with additional controls on the derivations of a sentence in the language of those grammars. Here, (i) V_T is the alphabet of terminal symbols, (ii) V_N is the alphabet of nonterminal symbols (we use $S \in V_n \setminus V_T$ as the start symbol), (iii) V_{PR} is the alphabet of a first-order predicate calculus PR and

$V_{PR} = \{T\} \cup CON \cup VAR \cup FUNC \cup PRED \cup \{\text{logical operators}\}$, where

$CON = \{\text{constant symbols}\}, VAR = \{\text{variable symbols}\}$,

$FUNC = \{\text{functional symbols}\} = FCON \cup FVAR$, where *FCON* are the constant functional symbols and *FVAR* are the variable functional symbols,

$PRED = \{\text{predicate symbols}\}$, (iv) *E* is an enumerable set referred to as the subject domain, (v) *H* is an interpretation of the PR calculus on *E*, (vi) *PARM* is a mapping from a symbol to a set of variable symbols, $PARM : V_T \cup V_N \rightarrow 2^{VAR}$, (vii) *L* is a finite set of labels, and (viii) R is a finite set of productions, which are 7-tuples of the form $\left(l, Q, A \to B, \pi_k, \pi_n, F_T, F_F\right)$ where $l \in L$,Q is a *well formed formula* (WFF) of PR (classical WFFs are logical sentences consisting of atoms and combinations of atoms using logical operations), being a condition of applicability of productions which contains only variables from VAR belonging to PARM(A), $A \to B$ is an expression (mapping) called the *kernel of production*, where $A \in V_N, B \in \left(V_T \cup V_N\right)^*, \pi_k$ a sequence of functional formulas corresponding to formal parameters of symbols from $\left(V_T \cup V_N\right), \pi_n$ a sequence of functional formulas corresponding to formal parameters of symbols from *FVAR* (non-kernel parameters), $F_T \subset L$, are labels permitted on the next step of derivation (to be defined below) if $Q = TRUE$, (permissible set in success), and $F_F \subset L$, are labels permitted on the next step of derivation if $Q = FALSE$ (permissible set in failure) (Stilman, 1993). Note: V^* denotes the letter monoid (under the operation of string concatenation in *V*) (i.e., the set of possible strings made from symbols in *V*) , the unit in V^* is the empty symbol ε, $V^+ = V^* \setminus \{\varepsilon\}$ and $|x|$ denotes the length of the string *x*.

Derivation results are formed from a finite set from V_T^* and formulas from π_n so that formal parameters of a terminal symbol (in V_T^*) are given a value from *E* and each symbol $f \in FVAR$ is matched with a mapping, say $h(f)$. In a control grammar *G*, one starts with a symbol *s* and has its parameters assigned maps $h(f)$ for all $f \in FVAR$.

The space L takes on the role of the set of initial permissible productions. One then applies productions from L as a symbol A, are applied to a current string ϑ, entering it in some placement. The newly constructed string, ϑ_A and a new permissible set are thus formed. Derivations for other strings obtained from a given one are independently formed thereafter. In the case of no productions from a permissible set being applicable, the derivation of the string is stopped. If this string only contains symbols in V_n (terminal symbols) , it then goes into the set of derivation results, otherwise it is discarded. Denote the set of derivation results starting with the frontier string *s* with respect to *G* as ${}_G\Delta(s)$.

Production application to the string ϑ, is applied in the following manner: one chooses the leftmost entry of the symbol *A* in the string ϑ. The production predicate *Q* is then calculated. If $Q = F$, then F_F becomes the permissible set and the process is halted. Otherwise, if $Q = T$, the symbol *A* is replaced by the string *B* and all formulas in π_k are computed using the parameters of the symbols of *B*; those parameters then assume the new values computed. The maps $h(f)$, for $f \in FVAR$, are computed by using the formulas from π_n and the permissible set becomes F_T, thereafter the production process is halted. If a formula from π_n does not change $h(f)$, it is omitted from the production record. Finally, a language generated by the controlled grammar *G*, denoted as $L[G]$, is the union of all sets that are derivation results of *G*, so that $L[G] = \bigcup_{x \in V_T^*} {}_G\Delta(x)$

For a particular state $\alpha \in \Gamma$ of an ABG Σ, trajectory patterns generated by Δ_Σ for a particular length *H*, form a language of trajectories for α given by $\Delta^H_{\Sigma,\alpha}$ and are composed specifically of the shortest and admissible degree 2 trajectories of length $l < H$ from Δ_Σ. Strings $\gamma \in \Delta^{H\,*}_{\Sigma,\alpha}$ formed from $\Delta^H_{\Sigma,\alpha}$, in turn, form the symbols for a *language of zones* $\Upsilon_{\Delta^H_{\Sigma,\alpha}}$ for the state α of Σ. This language of zones $\Upsilon_{\Delta^H_{\Sigma,\alpha}}$ then defines *LG zones* which, in turn, are the building blocks for defining *LG strategies* −strategies that define the survivability of pieces transferring from one location x_j, to a final location of interest x_t of X in the ABG Σ.

LG zones can be viewed as networks of trajectories on *X* that define the stratagem of the agents manipulating their pieces, (i.e., the sequence of strings representing agent action sequences). The set of LG zones is considered a higher level language while the language of trajectories is a lower level language in the scheme of defining viable paths for pieces. LG zones can also be represented as strings of symbols $h\left(p,\tau_p,t_{\tau_p}\right)\ldots h\left(\mathrm{q},\tau_q,t_{\tau_q}\right)$, each parameterized by the pieces $p \in P$, the trajectories for them, $\tau_p \in \mathrm{T}$, and time allocations for traversing those trajectories, t_{τ_p}. Furthermore, target states are formed based on large values of an overall value differential between agents such as $m(\alpha) = \left|\sum_{p \in P_1} v(p) - \sum_{q \in P_2} v(q)\right|$. If $p \in O$ (offboard) in state α, then $v(p) \approx 0$ and depends on the piece dynamics and rules of engagement for reentering the board from the offboard with some small nonzero value. Reachability criteria and sets of the board X of Σ are defined in the following manner: let $\chi_{x,p} : X \to Z^{+}$ define a map of the board X relative to the location x and piece p in Σ. Consider next a family of reachability sets from the point x, defined as:

$$M_{x,p}^{k} = \begin{Bmatrix} m \in P \mid R_p(\mathrm{x}, m) = 1, & k = 1 \\ m \in P \mid R_p(\mathrm{y}, m) = 1, y \in M_{x,p}^{k-1} \setminus \bigcup_{l<k} M_{x,p}^{l}, & k > 0 \end{Bmatrix} \tag{4.51}$$

Then define

$$\chi_{x,p}(y) = \begin{cases} k, & y \in M_{x,p}^{k}, \\ 2|P|, & y \in P \setminus \{\{x\} \cup M_{x,p}^{k}\}, \\ 0, & y \in \{x\} \bigcap \{P \setminus M_{x,p}^{k}\} \end{cases} \tag{4.52}$$

as the number of steps from x to y for piece p. $\chi_{x,p}$ is a symmetric distance and a metric on X for each p if R_p is, otherwise it is superadditive in X. One can symmeterize $\chi_{x,p}$ by defining an averaged (or weighted) symmetrization (Jensen-Shannon divergence): $\hat{\chi}_{x,p}(y) = \dfrac{\chi_{x,p}(y) + \chi_{y,p}(x)}{2}$, invoking a metric on X for each p. The ultimate result of the LG approach to constructing solutions for games is the following: the shortest trajectories from x to y of length l for a piece p on location x exist if and only if $\chi_{x,p}(y) = l$ and $l < 2|P|$. Furthermore, if R_p is symmetric, then all the shortest trajectories of $\mathrm{T}_p(x, y, l)$ can be generated by the control grammar of shortest trajectories of length $1, \Delta_{\Sigma,\alpha}^{1}$. If R_p is not symmetric, then the use of the symmetrization map $\hat{\chi}_{x,p}$ can define the condition above as $\hat{\chi}_{x,p}(y) = l$. The construction of these shortest trajectories implies the construction of the requisite LG strategies for Σ-based hypergames.

Trajectories are pruned by a criteria that defines a rule threshold such as smoothness in which the trajectory path from the start point to the end point changes direction minimally in the geometry of *X*. The model of strategies is formalized as a hierarchy of formal languages utilizing a class of generating grammars called the control grammars that employ semantics of the game in order to manipulate the strings of symbols that represent the game structurally. The geometry of the state space Γ is manifested by each state representing an ABG with the same board *X* and pieces *P* in different configurations.

Inceptions may generalize LG hypergames in the following manners: (i) multiple agents with co-opetitive characteristics so that an ABG Σ and Σ-induced hypergames may have *n* agents (and their hyperlinked inception level personas) with $P_i, i = 1, 2, ..., n$ mutually exclusive pieces respectively, and the sequence of starting and target states $\left\{S_0^i\right\}_{i=1,2,...,n}, \left\{S_t^i\right\}_{i=1,2,...,n}$ respectively, with *j*-wise combinations of draw states in payoffs, $\left\{S_{\left(i_1,...,i_j\right)}\right\}$ for all $j = 1, 2, ..., n$, (ii) the boards *X* may have non-classical geometries and topological properties of very general logic spaces such as quantum-gravity LQG spinfoams or superstring dimensions, and finally, (iii) the general stochastic and nonclassical probabilistic possibilities of a GTU-based constraint will form more exotic rule structures such as paraconsistent and modal logics generating non-classical trajectory languages for LG stratagems. LG hypergames can be redefined in terms of agents and their pieces and states of existence from different inception teams being hyperlinked to their different inception level persona and avatar existences, and (iv) quantum, quantum-gravity, and other exotic GTU-based grammars can replace the grammar-based approach of LG construction of strategies for corresponding quantum, quantum-gravity, and GTU-based strategies respectively. This would be an interesting new area of research for non-classical versions of LG grammars. This study earlier reviewed quantum pushdown automata as a model for each inception level. The corresponding quantum grammars shown initially in Crutchfield and Moore (1997) can be gleamed upon to form a basis for quantum LG grammars via the mechanisms of the LG construction.

Morphogenetic Approaches

Inceptions may involve massive numbers of agents depending on the social context of the game situation. For example, tactical warfare games may involve large numbers of agents and subgroups within the warring sides. Colonies of insects in co-opetition with other species for resources in a given landscape dwarf the sizes of typical human conflicts. In one of the most ostensible examples of emergent computation mimicking natural processes, ant colonization optimization (ACO) utilizing pheromone minimum cost paths on graphs, has become a viable large-scale multiagent computational tool for solving optimization problems (Dorigo and Stützle, 2004). The synergy between the local search power of natural ACOs and human reasoning leading to methods preventing local loops or optima, presented the more adaptive (simple) S-ACO algorithms for global searches. More generally, evolutionary computation has led to the development of algorithm classes that utilize this synergy to produce super-emergent spaces of adaptive processes, solving some combinatorially explosive problems. Morphogenesis, in particular, being the emergent phenomena of convergence of an organism base of DNA/RNA to well-formed organs and bodies, points to new computational power in the tradition of evolutionary processes and computation. Sheldrake (2009) and others have generalized this process to higher level fields of development called morphogenetic fields. Additionally, artificial logic systems have been axiomatically developed based on pure natural processes (Brenner, 2008).

In a seminal work on race segregation, Schelling (1969) developed a simulation using game-theoretic utilities and multi-agent models with cell neighborhoods and thresholds for happiness – happiness indices ($0 \leq \lambda \leq 1$) regarding an individual's propensity to be content with a proportion of like-minded or looking neighbors in their respective neighborhoods. In a recent update using Schelling's model of segregation, Barmpalias, Elwes, and Lewis-Pye (2013) refine what properties of local segregation present for global segregation patterns using Schelling's happiness threshold as a dynamic parameter (i.e., as $\lambda \rightarrow 1$). Counter-intuitive results were shown. For example, even for high happiness thresholds, global segregation patterns reach an upper limit. While the model in these studies is somewhat toyish since limited notions of happiness, neighborhood geometry (1-D arrangements of agents), and social interaction dynamics are used as well as limited periods of evolution, (i.e., as soon as a certain level of segregation is reached, the simulation is ceased), the results point to Markov process cellular multiagent models as viable patternizers for computational behavioral patterns. In particular, we may adopt this model with more realistic and general parameters spaces for games.

We are interested in the more general inception game and so two coalition types A and B (inception and inceptee teams) are used in place of race (or ethnicity). Instead of proximity locations (i.e., geographic neighborhoods), we develop a K-dimensional profile $\Theta_a = \left(\omega_i^a\right)_{i \in K}$ for an agent $a \in A \bigcup B$, consisting of risk profiles (risk aversion vs. aggressive on spectrum and with respect to risk value, payoff or utility intervals (could be thought of as income status, cost of real estate, net worth), resources (i.e., both internal and external - educational status, family wealth, network size, resilience,), and intangibles (non-decision making abilities). Happiness thresholds, λ are then the lower bound for an agent to accept being in a profile, where a proportion λ of agents within that same profile neighborhood (profile characteristic distance of μ) are from the same inception team. We may now define a general (discrete) probability divergence between two agent profiles, neighborhoods, and thresholds,

$$D\left(\Theta_{a_i} \middle\| \Theta_{a_j}\right) = \sum_{\omega} p_{a_i} \log \frac{p_{a_i}}{q_{a_j}} \tag{4.53}$$

where $\left(p_{a_i}, q_{a_j}\right)$ are the pdfs of the respective agent profiles. The sum is replaced by an appropriate integral in the case of continuous distributed profiles for agents. If the agents of an inception game are quantum-like, then an analogous version of (4.53) is the quantum relative entropy, $D_q\left(\Theta_{a_i} \middle\| \Theta_{a_j}\right) = \mathrm{tr}\left[p\left(\log p - \log q\right)\right]$, where p and q are the density operators on a Hilbert space for a_i and a_j respectively, and tr is the trace operator. The generalization of this divergence to quantum-gravity depends on one's approach either through LQG, superstring, or pregeometries. Entanglement of agents introduce the concepts of holographic and Wald entropies in superstring theory which would define a cross entropy leading to the analogous KL-divergence (Myers, Purhasan, and Smolkin, 2013). For LQG, a relative entropy may be constructed using the quantum causaloid approach of Hardy seen in more detail in Appendix D. In this study's most general approach, a divergence distance between two GTU constraints, G_l and G_k is defined by generalized GTU constraint distributions g_l and g_k respectively,

$$D_g\left(\Theta_{a_i}\middle\|\Theta_{a_j}\right)=\mathrm{tr}_g\left[g_i\left(\log\left(g_i\right)-\log\left(g_j\right)\right)\right] \tag{4.54}$$

where tr_g is a generalized trace operator acting on the space of GTU constraint distributions (Sepulveda, 2011). As before, a symmetrization of(4.54) is done by a weighted Jenson-Shannon divergence:

$$D_g^{\sigma}\left(\Theta_{a_i}\middle\|\Theta_{a_j}\right)=\sigma D_g^{s}\left(\Theta_{a_i}\middle\|\Theta_{a_j}\right)+\left(1-\sigma\right)D_g^{s}\left(\Theta_{a_j}\middle\|\Theta_{a_i}\right) \tag{4.55}$$

for $0\leq\sigma\leq1$. More general versions of divergences of which the KL-divergence is a special case of are available through the family of weighted Csiszar-Morimoto-Ali-Silvey *f*-divergences, of which weights in the divergence are defined by a twice differentiable convex function *f* with $f\left(1\right)=0$ and $f\,'\left(1\right)=0$ and weight function *w* (Csiszar, 1963; Kapur, 1994). In terms of our more general GTU structure, this divergence is defined as,

$$D_g^{f,w}\left(\Theta_{a_i}\middle\|\Theta_{a_j}\right)=\mathrm{tr}_g\left[wf\left(\frac{dg_i}{dg_j}\right)dg_j\right] \tag{4.56}$$

The Jenson-Shannon symmetrization ${}^{\sigma}D_g^{f,w}\left(\Theta_{a_i},\Theta_{a_j}\right)$, of (4.56) follows from (4.55).

When *f* is of the form; $f\left(\frac{dg_i}{dg_j}\right)=\left(\sqrt{\frac{dg_i}{d\nu}}-\sqrt{\frac{dg_j}{d\nu}}\right)^2$, where *v* is an absolutely continuous probability measure with respect to g_i and g_j, $\sqrt{D_g^f}$ is a metric on the space of pdfs defined on the agent profile space. One may then define metrizable geometries for computing profile neighborhoods.

Agent a_i is within μ neighborhood of agent a_j if $D\left(\Theta_{a_i}\middle\|\Theta_{a_j}\right)\le\mu$. A $\mu-nbhd$ of agent a is defined as $N_\mu(a)=\left\{\mathrm{c}\mid\mathrm{c}\in A\bigcup B, D\left(\Theta_a\middle\|\Theta_c\right)\le\mu\right\}$. Let $N=\left|A\bigcup B\right|$ be the total number of agents. Let β be the probability that $a\in A$. Agent a will be $(\mu,\lambda)-happy$ if $\frac{\left|N_\mu(a)\right|}{N}\ge\lambda$. If agent a is not $(\mu,\lambda)-happy$, then it may exchange teams with an agent *b* from the other team who is also not $(\mu,\lambda)-happy$ This exchange rule assumes that the threshold criteria has the same intrinsic value for each agent. Furthermore, each agent may have different happiness thresholds λ_a. One can define an exchangeable tolerance mapping between happiness indices between agent. In this way a mapping from one threshold value λ_a for agent a to another λ_b for agent b are *happiness equivalent*. In the same manner that pain thresholds may be standardized to a certain degree, happiness equivalence is a homogenizing transformation of happiness from one agent to another (pairwise). Multi-wise equivalence becomes more complex since a triangularization must occur between agent happiness thresholds (i.e., happiness thresholds will differ in equivalence depending on the participating equivocating group of agents). Let $f_a : \lambda_a \to (\lambda_{-a})$ be such a mapping for an agent a from its happiness threshold to the series of thresholds for all other agents $-a$. Then an agent $a \in A$ who is not $(\mu, \lambda)-$happy will exchange teams with an agent $b \in B$ who is not $(\mu, f_a(\lambda_a))-$happy.

Patterns of defection or in the case of Schelling, neighborhood segregation, can be seen as agent movement between teams and hence of inception game values and payoffs since agent defection (including all *n*-agencies ending in a final defection to one team or another where 0-agency means no defection or total non-covert loyalty) will mean a shift in social power as defined earlier for inception games. If total or nearly total segregation or team defection is reached, then the inception becomes inevitable or inevitably prevented, depending on the team agents doing the dominant defection. The significance of Barmpalias, Elwes and Lewis-Pye's study to inceptions is that, at least in the simple 1-D profile (geographic proximity) case for neighborhoods, there arose five regions of behavior: for κ a solution of $\left(\frac{1}{2} - \kappa\right)^{1-2\kappa} = (1-\kappa)^{2-2\kappa}$, the regions are (i) $\lambda < \kappa$, (ii) $\lambda = \kappa$, (iii) $\kappa < \lambda < \frac{1}{2}$, (iv) $\lambda = \frac{1}{2}$, and (v) $\lambda > \frac{1}{2}$. For sufficiently large number of agents n, $n \gg \mu$, some counterintuitive patterns emergent. For example, in region $\{\lambda < \kappa\}$, if

$\varepsilon > 0$, then for sufficiently large μ, the probability of a randomly chosen unhappy agent swap is $< \varepsilon$. On the other hand, in the region $\left\{ \lambda \in \left(\kappa, \frac{1}{2} \right) \right\}$, there exists a constant d such that $\forall \mu, n\left(n \gg \mu\right)$, the probability that a randomly chosen agent is in a run of swaps of length $\geq e^{\frac{w}{d}}$, in a final configuration, is $> 1-\varepsilon$. In region $\left\{ \lambda = \frac{1}{2} \right\}$, there exists a constant $c < 1$ such that $\forall \delta > 0$, the probability that an agent chosen uniformly randomly will belong to a run of swaps that is $> \delta\lambda^2$, in the final configuration, is $\leq c^{\delta}$. In region $\left\{ \lambda > \frac{1}{2} \right\}$, for sufficiently large μ that $\lambda > \frac{\mu+1}{2\mu+1}$, then with probability $\to 1$, as $n \to \infty$, for a given initial configuration, the event of complete segregation eventually occurring has probability 1. In the last region, if the happiness index dictates at least a majority of like-agents within a large enough defined profile neighborhood, complete segregation eventually occurs with probability 1. In an inception this is equated with eventual social power segregation and little if no inception penetration or coercion, (i.e., sufficient integration arising from initially small segregation happiness with $\lambda \to \frac{1}{2}^{+}$ and $n \gg \mu \to \infty$ means essentially that the probability of no inception converges to 1). So, this may be considered a situation in which well intentioned convergence to harmonization with the caveat of an implied perpetual small majority eventually still produces no inception. The only viable condition on happiness for potential inception (integration), is the explicit condition of giving up the majority position. This begs the question, “why would an agent relinquish the intent (happiness) of local common pluralism in the midst of covert inception intentions?” This is the most natural psychological profile of *n*-agencies, eventually working among like agents with the exception of existentially being on the opposite team.

Game Categories and Topoi

All games may be expressed in a more powerful and abstract form known as categories and topoi. Topoi generalize the concept of point-sets topologies and are specializations of categories which, in turn, generalize mathematical objects, a higher order mathematical abstraction alternative to model theory constructs and mathematical logic (Lawvere & Schanuel, 1997). Appendix E gives a brief review of categories and topoi. Here, we closely follow Vanucci (2004) in reviewing general game forms that will be expressed as categories of Chu spaces as the most general mathematical object for games. For brevity and immediate relevancy, we will discuss coalitional game formats only as inceptions are being presented a coalition games, albeit recursive coalition games.

Def. An *effectivity coalition game* form (*ECGF*) is a tuple $G=(I,P,E)$ where I is the index set of agents, P is the set of payoff (outcome) functors, and $E:2^I \to 2^{2^I}$ is a generalized effectivity functor (*GEF*).

*GEF*s map subsets of agents to subsets or collections of subsets of agents and satisfy the conditions, (i) $E(\varnothing)=\varnothing$, (ii) $\varnothing \notin S$ for any $S \subseteq I$, (iii) $P \in E(S)$ for any

$S \subseteq I, S \neq \varnothing$, (iv) $E(I) = 2^P \setminus \{\varnothing\}$. More general forms of *ECGF*s are given by social situation forms,

Def. A social situation form (*SSF*) is a tuple $G = (I, P, \mathrm{M})$ where *I* is the index set of agents, P is the set of payoff (outcome) functors, and $\mathrm{M} \subseteq \mathrm{K}(I,P) \times 2^I \times P \times \mathrm{K}(I,P)$ where $\mathrm{K}(I,P) = 2^I \times 2^P$ denotes the set of position forms on (I,P). Furthermore, $\mathrm{M} \subseteq \{((G,A),U,x,(T,B)) : x \in A, U \subseteq S, U \subseteq T\}$ denotes the inducement correspondence (Greenberg, 1990).

The set of *GEF*s are contained (specializations of) in the set of *SSF*s. We now define Chu spaces as they will be the most general form of spaces that will describe (coalition) games by categories.

Defs. Let A be a category with finite products, $K \in Ob(A)$ (categorical objects of *A*). A *Chu space over K* is defined as the tuple $\mathcal{A} = (X, Y, \ )$ where $X, Y \in Ob(A)$ and $\in \hom_A (X, Y, K)$. Additionally, a *Chu-transform* from a Chu space $\mathcal{A} = (X, Y, \ )$ to another Chu space $\mathcal{A}^* = (X^*, Y^*, {}^*)$ is a pair of functors (*A*-morphisms),

$f \in \hom_A(X, X^*), \mathrm{g} \in \hom_A(\mathrm{Y}, \mathrm{Y}^*),\ \varphi = (f, g)$ such that ${}^* \circ (f \times \mathrm{id}_{Y^*}) = \circ(\mathrm{id}_X \times g)$.

Results. Let A be a category with finite products and $K \in Ob(A)$. Let $\mathrm{Chu}(A,K) = (O, \mathrm{hom}, id, \circ)$ where O is the class of all Chu spaces (induced from A), $\mathrm{hom}(\mathcal{A}, \mathcal{A}^*)$ is the set of all Chu transforms from $\mathcal{A}$ to $\mathcal{A}^*$ for any $\mathcal{A}, \mathcal{A}^* \in O$, $\mathrm{id}_{\mathcal{A}} = (\mathrm{id}_X^A, \mathrm{id}_Y^A)$ for each $\mathcal{A} = (X, Y, \ ) \in O$ and $\varphi^* \circ \varphi = (f^* \circ_{\mathcal{A}} f, g^* \circ_{\mathcal{A}} g)$ for any pair of Chu transforms $\varphi = (f, g) \in \mathrm{hom}(\mathcal{A}, \mathcal{A}^*)$ and $\varphi^* = (f^*, g^*) \in \mathrm{hom}(\mathcal{A}^*, \mathcal{A})$, then $\mathrm{Chu}(A,K)$ is a category. In particular, let Poset denote the category having the class of all partially ordered sets as objects and the class of order-homomorphisms (antisymmetric, transitive, and reflexive binary relation) as morphisms, K a set and $\hat{K} = (K, =)$. Then $\mathrm{Chu}(\mathrm{Poset}, \hat{K})$ is a category.

Finally, we arrive at the category representation of games in the form of *ECGF*s.

Result. Let $\mathrm{ECGF} = (\mathrm{Ecgf}, \mathrm{hom}_{\mathrm{ECGF}}, id, \circ)$ where Ecgf is the class of all *ECGF* games, and for any $e = (I, P, E)$, $e^* = (\mathrm{M}, \mathrm{Q}, E^*) \in Ecgf$,

$$\mathrm{hom}_{\mathrm{ECGF}}(e, e^*) = \left\{ \begin{array}{l} (f, g) : f \in \mathrm{hom}_{\mathrm{Poset}}\left((2^I, \supseteq), (2^M, \supseteq)\right), \\ \qquad g \in \mathrm{hom}_{\mathrm{Poset}}\left((2^Q, \supseteq), (2^P, \supseteq)\right) \mid \text{for each } S \subseteq I \text{ and} \mathrm{B} \subseteq Q \\ \qquad B \in e^*(f(S)) \Leftrightarrow g(B) \in e(S) \end{array} \right\}$$

then ECFG is a full subcategory of $\mathrm{Chu}(\mathrm{Poset}, 2)$. Here 2 represents the Boolean set

$\{0,1\}$.

Conjecture. Let $\mathrm{SSF}=\left(\mathrm{Ssf},\mathrm{hom}_{SSF},id,\circ\right)$ where Ssf is the class of all *SSF* games, and for any $e=\left(I,P,\mathrm{M}\right),e^{*}=\left(K,Q,\mathrm{M}^{*}\right)\in SSF$,

$$\mathrm{hom}_{\mathrm{SSF}}\left(e,e^{*}\right)=\left\{\begin{array}{l}(f,g): f\in\mathrm{hom}_{\mathrm{Poset}}\left(\left(2^{I},\supseteq\right),\left(2^{K},\supseteq\right)\right),\\ \qquad g\in\mathrm{hom}_{\mathrm{Poset}}\left(\left(2^{Q},\supseteq\right),\left(2^{P},\supseteq\right)\right)\mid \text{for each } S\subseteq I \text{ and} \mathrm{B}\subseteq Q\\ \qquad B\in e^{*}\left(f\left(S\right)\right)\Leftrightarrow g\left(B\right)\in e\left(S\right)\end{array}\right\}$$

then SSF is a full subcategory of $\mathrm{Chu}\left(\mathrm{Poset},2\right)$.

Coalition games in ECFG or SSF form can then be classified as full subcategories of the Chu space category $\mathrm{Chu}\left(\mathrm{Poset},2\right)$ induced by the set of partially ordered sets. In particular , an inception game is a (recursive) SSF with social structure as depicted by a GGT game. Each recursive level in inceptions is a coalition subgame. Hence, we may apply the subcategory mapping to $\mathrm{Chu}\left(\mathrm{Poset},2\right)$ for each recursion level.

Recall that an inception game was eventually expressed as a generalized GTU-based recursion game $\Gamma=\left\{\left\{\Gamma_{i}\right\}_{i\in L},H,A,S,G,G_{\Gamma}\right\}$. Then Γ may be represented as a recursion or series of subcategories of $\mathrm{Chu}\left(\mathrm{Poset},2\right)$ or a generalized set category based on $\mathrm{Chu}\left(\mathrm{Poset},2\right)$ given by an *n*-category which is induced or enriched *n* times by $\mathrm{Chu}\left(\mathrm{Poset},2\right)$ and labeled $C_{n}\left(\Gamma\right)=n\mathrm{Chu}\left(\mathrm{Poset},2\right)$ where *n* is the number of levels generated in an inception game See Appendix E for a review of higher order categories via *n*-categories and their generalization *(n,r)*-categories. If, on the other hand, Γ is defined as a simple Cartesian product of *n* copies of $\mathrm{Chu}(\mathrm{Poset},2)$, then the product category $C_{PR}(\Gamma) = \bigotimes_{n} \mathrm{Chu}(\mathrm{Poset},2)$ endowed with component-wise projective mapping from Γ to each copy of $\mathrm{Chu}(\mathrm{Poset},2)$ representing an inception level is the appropriate category. See the end of Appendix E for details on the category product. In order to convert these categories to topoi, a truer representation of generalized point sets and proper set operations must be present. In particular by satisfying the conditions of Giraud's axioms for a category, a Grothendieck topos is created (Giraud, 1972). Hence, either $C_{PR}(\Gamma)$ or $C_h(\Gamma)$ may then be converted to a Grothendieck topos by the imposition of those conditions given in Giraud (1972).

The category-theoretic structure of $C_{PR}(\Gamma)$ or $C_h(\Gamma)$ is essentially morphism-free. No mappings or transofmrations from one game object to another were introduced into these game categories. How does one, in general, tranmsform from one game structure to another, in an invariant manner, (i.e., games that can be transformed to each other without loss of information, strategy formation, or dynamics)? Games are inherently highly dynamic and computationally complex for larger numbers of agents, strategies, payoff dynamics, and state dynamics. In the case of the topology of 2x2 games of Robinson and Goforth, these transformations may be envisioned as swaps on game colored tiled squares. Swaps are transformations based on game components such as payoff families, equilibria families, symmetry among agents, and number of dominant strategies. Morphisms can then be based on these spaces of game components. Hence, a more general game morphism caould be reflected by a multi-dimensional morphism from the space of payoff outcomes for agents, dominant strategies, equilibria, and symmetry of agents. Half swaps are those swaps that conclude in ties between games as payoff resultant preferences. These morphisms essentially induce orderings of games. Game graphs can then be developed based on these game order preferences (Bruns, 2010). The case of game morphisms will larger games with complex dynamics and interactions is far from generalized from the 2x2 game spaces. When larger games can be partitioned into linked 2x2 subgames, subgame perfect strategies strengthen a kind of space of linked morphisms where each full game morphism can be decomposed into smaller 2x2 subgane morphisms as discussed above.

Appendix C: Zadeh's Generalized Theory of Uncertainty

Zadeh (2006) has given a proposal for a generalized theory of uncertainty (GTU) in which notions of uncertainty including: (a) probabilistic, (b) possibilistic, (c) veristic, (d) usuality (fuzzy probability), (e) random, (f) fuzzy graphic, (g) bimodal, and (h) group types of uncertainty are modeled through a generalized constraint model. Complementary to this, a generalized constraint language (*GCL*) consists of all generalized constraints coupled with the rules for qualification, combination, and propagation. A generalized constraint (*GC*) is a triplet of the form (X, r, R) where X is a constrained variable, R is a constraining relation, and r is an indexing variable which identifies the modality or type of constraint semantics. The index list consists of the following pneumonic: $r = blank$, possibilistic, $r = p$, probabilistic, $r = v$, veristic, $r = u$, usuality, $r = rs$, random set, $r = fg$, fuzzy graph, $r = bm$,bimodal, and $r = g$, group variable. A formal uncertainty language such as a *GCL* calculates precisiations (the mapping of a vague measure into a precise number) more readily than formalized logics. Constrained variables, R can take the form of: (a) a general m-vector, (b) a proposition, (c) a function, (d) a function of another variable, (e) a conditioned variable, (f) a structure, (g) a group variable, or (h) another generalized constraint. Bi-valent conjunction, projection, and propagation operators, $\otimes_c, \otimes_{proj}, \otimes_{prop}$ respectively act on two (possibly different) *GC* objects, $(X_{k_1}\ is\_i_1\ R_{j_1}), (X_{k_2}\ is\_i_2\ R_{j_2})$ to generate a third (possibly different) GC object $(X_{k_3}\ is\_i_3\ R_{j_3})$.

A *GC* object, $g = (X, r, R)$, is associated with a test-score $ts_g(u)$ which associates an object *u* (which the constraint is applicable to), a degree to which *u* satisfies the constraint. The test score defines the semantics of the constraint that is associated with *g*. The value of $ts_g(u)$ may be a point in the unit interval, $[0,1]$, a vector, or other mathematical structure such as a member of a semi-ring, lattice, poset, or bimodal distribution. The relation, *R* from *g* is allowed to be nonbivalent, as in a fuzzy equivalence. In this way, a *GC* generalizes a fuzzy set and so, a *GCL* can lead to a generalized fuzzy system of generalized constraints.

Zadeh presents a precisiation natural language (*PNL*) as a means of assigning precise meaning to a proposition drawn from a natural language (*NL*) through a GC. The *PNL* is then a mapping, $\Gamma_{PNL} : p \rightarrow g$ from a proposition, *p* to a *GC*, $g = (X, r, R)$. Hence, information, in general, is representable as a *GC* because a proposition is a carrier of information, being a potential answer to a question. Let *S* be a system. Let S_{P} be the space of all propositions in *S*, S_{Υ} the space of *GC*s in *S*, and $\Gamma_{PNL}(S)$, the mapping assigned to a *PNL* for *S*. Then $\Gamma_{PNL}(p) \in S_{\Upsilon}$ for a precisiable proposition *p* in *S*. Denote the space of precisiable propositions of *S* by $S_{\mathrm{P}}^{'}$. In general, $S_{\mathrm{P}}^{'} \subset S_{\mathrm{P}}$ for *NL* systems. Let S_{GCL} be the space of all *GC*s of *S*. Then S_{GCL} is more expressible relative to *S*, than a first order logic, modal logic, Prolog, and LISP is to *S*, if *S* is a *NL*. Because quantum logics can be derivable as a family of subsets of fuzzy probability structures (using Lukasiewicz operators), a *GCL* can be formalized for it, though quantum probability may be framed as a generalized probability theory as well (Cohn, 2007). The importance of this is that any quantum logic (logical system), *L*, with an ordering set of probability measures, *S*, are isomorphic (representable) in the form of a family of fuzzy subsets of *S*, $\mathcal{L}(S)$, satisfying certain conditions, including the use of Lukasiewicz operators instead of Zadeh's operators on fuzzy sets (Pykacz, 2007). Hence *L* is representable by a *GCL*. In this sense, quantum logics are special cases of (and isomorphic to) fuzzy probability logics and so are in the realm of a *GCL* representation.

Does this *GTU* represent a kind of generalized logic in the taxonomy of algebraic logics? In other words, can the GTU transcend a spectrum of the algebraic hierarchy of logics, which include fuzzy and quantum logics, and of other duals to these, notably referred to as dual-intuitionistic logics or paraconsistent logics? Paraconsistent logics are logic systems that formalize inconsistent nontrivial logics in the sense of the rejection of the principle of explosion (noncontradiction), the premise that anything follows from contradictory premises (Béziau, 2000). The principle of explosion is as follows: for a proposition *p*, and an arbitrary claim *A*,

$p \wedge \neg p$ (premise)
p (conjunctive elimination)
$p \vee A$ (weakening for any A)
$\neg p$ (conjunctive elimination)
A (disjunctive syllogism)
$\Rightarrow A$ (conclusion)

Paraconsistent logics overcome Gödelian limitations, i.e., incompleteness of axiomatic systems. In addition, because they are accepting of the truth or falsehood of both a premise and its negation, they are flexible in overcoming other seemingly paradoxical physical theories such as the quantum nature of long range gravitational influences or macroscopic and mesoscopic entities.

Rough set theory is a concept of the uncertainty in the coarseness of sets. Whereas, fuzzy sets discern uncertainty in the vagueness of sets. While fuzzy set uncertainty is expressible in algebraic logics – Heyting algebras – and hence expressible as PNLs, rough set approximation spaces can be expressed as Pawlak-Brouwer-Zadeh lattices which can be expressed as types of distributive De Morgan lattices (Cattaneo & Ciucci, 2002; Dai, Chen, & Pan, 2006; Greco, Matarazzo, & Slowinski, 2011). These lattices can then be expressed as logical algebras (Heyting Wajsberg algebras) with PNLs. The corresponding generalized constraint $g_R = \left(X, r_g, R_g\right)$ where r_g is the placeholder index for rough set uncertainty, can then be categorized as a GTU representation of rough set uncertainty (with respect to set boundary approximations).

Dempster-Shafer belief function theory first derived from a series of papers, Dempster (1967), Shafer(1976), and Schum (1994) and the intuitionist probability theory as detailed in Narens (2003) are considered bimodal distributions of probabilistic type and are categorized under the index *bm*. The details of the space of the *bm* index of uncertainty distributions fall under three subtypes and are given in Zadeh (2006).

Appendix D: Causaloids and Quantum-gravity Machines

Causaloids are operationally constructed as a potential means of building a physical theory encompassing both the probabilistic calculus of quantum theory (QT) and the indefinite causal structure of general relativity (GR). The treatise from Lucien Hardy in a series of papers will be followed closely in this discussion (Hardy, 2005, 2007, 2008, 2008b). Causaloids are an attempt at building a framework for construction of a mathematical physical theory that correlates with recorded data, while handling situations when an indefinite causal structure is present or when a time sequential evolution is not. In GR, causal structure is dynamic because of the nature of the spacetime metric and its dependence on the gravitational force from the distribution of mass. In QT, if time cannot be handled sequentially in the evolution equations, quantum uncertainty ensues. Any theory of quantum gravity (QG) must then, in all likelihood, be capable of handling indefinite causal structures while retaining a consistent probabilistic calculus.

To this end, consider two spacetime regions in the universe given by R_1 and R_2 which are spatio-temporally disconnected. One would like to posit a probabilistic statement about the region R_1 conditional on information from R_2. The current approaches to QG via path histories, such as LQG spinfoams, M-Theory, evolutional equations, or local infinitesimal changes via differential equations fall short in this scenario because of the disconnect in spacetime probabilities. Spacelike separate regions dictate that correlational operators should be tensor products, $A \otimes B$ whereas, temporal-sequential regions should use direct products, AB. For the causaloid formalism, a third kind of product, called the question mark product will be introduced. It is given by the notation, $[D?B]C = DCB$ where C is a causal switch operator that indicates either the tensor $\otimes$ or direct product (blank) depending on the causal structure of the regions. Note that ? remains a linear operator. Which product to use is therefore dictated by the causal structure of $R_1 \cup R_2$. In the case of probabilistic theories, such as quantum probability, with adjacent causality, a reduction in the information needed to infer the state of compound systems is present due to correlation relationships. This is referred to as *second 1evel physical compression*. The fundamental question that causaloids attempt to answer are the probabilistic propositions of the form:

$$p(X_{R_1} \mid F_{R_1}, X_{R_2}, F_{R_2}) \tag{5.1}$$

where X_{R_i} is an observed measurement (observable) of a physical entity made in R_i and F_{R_i} is some action performed in R_i., i.e., some control parameterization of the measurement device.

A topological assumption is made about spacetime regions *R*. Each region *R* may consist of the union of many elementary regions and composite regions consisting of more than two elementary regions themselves, $\{R_i\}$. An elementary region in spacetime is a simple region that may not be operationally reduced in terms of the measurement devices. Let Υ denote the space of elementary regions in a spacetime universe. For the purposes of this paper, Υ may be planck-scale cells or pixels in a discrete LQG-spinfoam or planck-scale computer (PSC). To standardize operations on Υ, attach to each region *R*, a set of vectors (operators), $r_{(x_R, F_R)}(R)$ and define the causaloid product, $\otimes^{\Lambda}$ by:

$$r_{(X_{R_i} \cup X_{R_j}, F_{R_i} \cup F_{R_j})}(R_i \cup R_j) = r_{(X_{R_i}, F_{R_i})}(R_i) \otimes^{\Lambda} r_{(X_{R_j}, F_{R_j})}(R_j) \tag{5.2}$$

For a composite region, $R = \bigcup_i R_i$, through the causaloid product, $\otimes^{\Lambda}$, the r vectors of the elementary regions would built the r vectors for the composite. Crucially, $p(X_{R_1} \mid F_{R_1}, X_{R_2}, F_{R_2})$ is well-defined $\Leftrightarrow v \parallel u$ where:

$$\begin{aligned} v &\equiv r_{(X_{R_i}, F_{R_i})}(R_i) \otimes^{\Lambda} r_{(X_{R_j}, F_{R_j})}(R_j) \\ u &\equiv \sum_{Y_{R_j}} r_{(Y_{R_i}, F_{R_i})}(R_i) \otimes^{\Lambda} r_{(X_{R_j}, F_{R_j})}(R_j) \end{aligned} \tag{5.3}$$

and the sum is over all possible observations Y_{R_j} made in R_i, consistent with the action F_{R_i}. Here,

$$p(X_{R_1} \mid F_{R_1}, X_{R_2}, F_{R_2}) = \frac{|v|}{|u|} \tag{5.4}$$

Now consider the collection of data that will be utilized to form a probability statement about the regions. Let the data be a collection of triplets (x, F_x, s_x), where x is the location of the observable, F_x is a parameterization (knob control setting) of the measurement operator (apparatus), and s_x is the outcome of the measurement. Next, consider a temporal manifestation of data collection via a series of probes in space. Let the quadruple $d_{i,n} = \left\{ \left(t_i, \left\{ t_i^{n,m} \right\} \right), n, F_{n,x}, s_{n,x} \right\}_{i,n}$ be the collection of data made by probe n at time t_i of the observable at location x with outcome $s_{n,x}$, using controls $F_{n,x}$. The series $\left\{ t_i^{n,m} \right\}$ represent the time delays seen by probe n of the results from m other probes. For each time slot t_i and probe, n, $d_{i,n}$ is recorded. At the end of the experiment, the series

$\{d_{i,n}\}_{i,n}, i = 0,1,..., N$ would have been recorded. Now consider a repeated experiment in which several controls, $\{F^{e}_{n,x}\}, e = 1,2,..., E$ are used where E is the number of experiments performed. This would be cosmologically problematic, but there are viable alternative setups to this thought experiment. Before showing this, we procede to define the structure of a causaloid which will determine the causaloid product $\otimes^{\Lambda}$ and r vectors for regions. The series of data, $\{d_{i,n}\}_{i,n}, i = 0,1,..., N$, which Hardy refers to as card stacks, one card per $d_{i,n}$, is divided into those which are consistent with a particular parameterization, F. For simplicity, this subset of cards is denoted by F. For any particular run of the experiment, say X, then $X \subset F \subset V$ where V denotes all possible cards (experiments).

Denote R_O to be the region specified by the set of cards in V consistent with the condition $x \in O$ (measurements in O). Let R_x be the elementary region consisting only of the cards in V with x. Regions are then spacetime entites where local choices for measurement (action) are taken. With this understanding, the term X_{R_o} means $X \cap R_o$, that is, the cards from a run stack X that belong to the region R_O. Define the procedure or action $F_{R_O} = F \cap R_O$, as the cards from F that belong to R_O. The pair (X_{R_O}, F_{R_O}) now defines the measurement result and action taking place in the region R_O. For notation sake, one can label the observations taking place in R_O as $Y_{R_O} = Y \cap R_O$. One now returns to the fundamental problem of calculating the probabilistic propositions given by (5.1).

Since this is a probabilistic statement, one may inject a Fisherian (frequentist), Kolmogorovian (axiomatic calculus), Bayesian (conditioning calculus), or other notions of probability calculus in these definitions over regions. The point of departure for this paper would be to inject a more general approach to intuition and information transfer, that is, a notion of generalized fuzzy logic from GTU (Zadeh, 2005). For the purpose of brevity in this review, the more powerful version of a causaloid, the universal causaloid will be constructed here. In this particular version of a causaloid framework, repeating experiments will not be necessary for the inference needed to calculate probabilistic propositions.

In classical statistical approaches, repeating experiments are the calculus for constructing robust estimators of the parameters of the underlying probability densities or constructs of the phenomena under investigated. However, in the environment of the universe, resetting the clock to repeat the experiment of the probing bodies illustrated before as the means of data collection is problematic. In this review of causoloids, two categories will be viewed. The first will be with respect to repeated trials of measurements. The second will be a notion of universal causaloids where repeated experiments are not taken. Instead a larger deck of observations will be made and the metric for measuring the truth of a probabilistic proposition will be changed to approximate truth. The first kind of causaloid will be reviewed first. Consider two composite regions of spacetime, R_1 and R_2 with corresponding experiment controls (procedures), F_1 and F_2. Probabilistic statements (propositions) of the form:

$$p(Y_2 \mid Y_1, F_2, F_1) \tag{5.5}$$

will be the center of inquiry for causaloid frameworks. This is simply the probability of observing the outcome Y_2 using procedure F_2 in region R_2 given that Y_1 was observed using F_1 in region R_1. Statistically, this is a likelihood function. However, because the regions involved may be spatio-temporally vastly separated with no ordered or connected causal structure, its calculation would not be well defined. A deeper and more general formulation must be developed for such physical cases. One must then find if a proposition is well defined (w.d.) and if so, find out how to calculate it.

Consider a sufficiently large region, R covering most of V. Next, assume that some C is a universal condition on the procedures, $F_{V\setminus R}$ and outcomes, $Y_{V\setminus R}$ respectively in the region, $V \setminus R$ such that the probabilities, $p(Y_R \mid F_R, C)$ are w.d. This guarantees the existence of these likelihoods in a sufficiently large portion of the computable universe. Assuming the existence of C, the likelihood functions will simply be abbreviated as $p(Y_R \mid F_R)$ and are w.d. Applying reductionism to this large region, three kinds of physical compressions will be defined that will help in forming the calculations for the likelihood computations. First level compression will apply to single regions. Second level compression will apply to composite regions. Finally, third level compression will be applied to matrix constructs that are manifested out of calculations pertaining to first and second compressions. Define a shorthand for likelihoods, using the notation, $\alpha_1 = (Y_{R_1}, F_{R_1})$ for each possible pair in the region R_1:

$$p_{\alpha_1} = p(Y_{R_1}^{\alpha_1} \cup Y_{R\setminus R_1} \mid F_{R_1}^{\alpha_1} \cup F_{R\setminus R_1}) \tag{5.6}$$

In a physical theory that is governed in part by a probability calculus, the set of possible p_{α_1} can be reduced in size by relations, so that a minimal vector of p_{α_1} suffices in expressing itself and without loss of generality, in a linear relationship,

$$p_{\alpha_1} = \bar{r}_{\alpha_1}(R_1) \cdot \bar{p}(R_1) \qquad (5.7)$$

where the state vector, $\bar{p}(R_1)$ is given by a minimal index set, Ω_1 :

$$\bar{p}(R_1) = \begin{pmatrix} \cdot \\ \cdot \\ p_{l_i} \\ \cdot \\ \cdot \end{pmatrix}, \; l_i \in \Omega_1 \qquad (5.8)$$

Ω_1 is referred to as the fudicial set of measurement outcomes. Since in a probability manifold, probabilities are linear, a linear relationship in the above compression is most efficient. Ω_1 may not be unique in general, but since it defines a minimal set, there exist a set of $|\Omega_1|$ linearly independent states in $\bar{p}$. The first level compression for region R_1 is then represented by the matrix:

$$\Lambda_{\alpha_1}^{l_1} \equiv \left(r_{l_i}^{\alpha_1} \right) \qquad (5.9)$$

where $r_{l_i}^{\alpha_1}$ is the l_i^{th} element of the vector $\bar{r}_{\alpha_1}$. Compression in the matrix $\Lambda_{\alpha_1}^{l_1}$ is manifested by the degree of rectangularity (lack of squareness), a flattening of the matrix.

Second level compression is shown for composite regions. Consider two regions R_1 and R_2. Form the composite region, $R_1 \cup R_2$ and express its state:

$$\bar{p}(R_1 \cup R_2) = \begin{pmatrix} . \\ . \\ p_{k_i k_j} \\ . \\ . \end{pmatrix}, \; k_i k_j \in \Omega_{12} \tag{5.10}$$

It has been shown that the second level fiducial set, Ω_{12} can be chosen such that

$\Omega_{12} \subseteq \Omega_1 \times \Omega_2$ (cartesian product). Further, one can express the likelihoods as:

$$\begin{aligned} p_{\alpha_1\alpha_2} &= \bar{r}_{\alpha_1\alpha_2}\left(R_1 \cup R_2\right) \cdot \bar{p}\left(R_1 \cup R_2\right) \\ &= \sum_{l_i l} r_{l_i}^{\alpha_1} r_{l_j}^{\alpha_2} p_{l_i l_j} \\ &= \sum_{l_i l} r_{l_i}^{\alpha_1} r_{l_j}^{\alpha_2} \bar{r}_{l_i l_j} \cdot \bar{p}\left(R_1 \cup R_2\right) \end{aligned} \tag{5.11}$$

Then the following must hold:

$$\bar{r}_{\alpha_1\alpha_2}\left(R_1 \cup R_2\right) = \sum_{l_i l_j} r_{l_i}^{\alpha_1} r_{l_j}^{\alpha_2} \bar{r}_{l_i l_j}\left(R_1 \cup R_2\right) \tag{5.12}$$

since there exist a spanning set of linearly independent state elements in $\bar{p}\left(R_1 \cup R_2\right)$.

Now define the matrix representation for second level compression of $R_1 \cup R_2$.

Let

$$\Lambda_{l_i l_j}^{k_1 k_2} = \left(r_{k_i}^{l_1} r_{k_j}^{l_2}\right) \tag{5.13}$$

where $r_{k_i}^{l_1} r_{k_j}^{l_2}$ is the $k_i k_j{}^{th}$ element of the vector $\bar{r}_{l_i l_j}$. One can then express these components as:

$$r_{k_i k_j}^{\alpha_1\alpha_2} = \sum_{l_i l_j} r_{l_i}^{\alpha_1} r_{l_j}^{\alpha_2} \Lambda_{l_i l_j}^{k_i k_j} \tag{5.14}$$

and in this way calculate the likelihoods for the composite region from those of each of its constituent component regions. This is the second level compression above and beyond first level compression of simple regions for the case of a composite region. One using this definition of second level compression to define the causaloid product, $\otimes^{\Lambda}$:

$$\bar{r}_{\alpha_1\alpha_2}\left(R_1 \cup R_2\right) = \bar{r}_{\alpha_1}\left(R_1\right) \otimes^{\Lambda} \bar{r}_{\alpha_2}\left(R_2\right) \tag{5.15}$$

Because this definition generalizes completely to higher level composite regions, the second level compression matrices are defined analogously for *n*-region composites by:

$$\Lambda^{k_1 k_2 \ldots k_n}_{l_1 l_2 \ldots l_n} \tag{5.16}$$

Now consider a master matrix that consists of all levels of lambda matrices for elementary regions, R_x, for a set x, where O_R is the set of x in the region R:

$$\begin{pmatrix} \Lambda^{\alpha_x}_{l_1 x_1} & \forall x_1 \in O_R \\ \Lambda^{k_{x_1} k_{x_2}}_{l_{x_1} l_{x_2}} & \forall x_1, x_2 \in O_R \\ \Lambda^{k_{x_1} k_{x_2} k_{x_3}}_{l_{x_1} l_{x_2} l_{x_3}} & \qquad \forall x_1, x_2, x_3 \in O_R \\ . \ldots\ldots\ldots\ldots\ldots & \\ . \ldots\ldots\ldots\ldots\ldots & \end{pmatrix} \tag{5.17}$$

In a consistent probabilistic formalism for a phsyical theory, these Λ-matrices will in turn have a relationship among each other. These relationships, along with a rule set for calculating other Λ-matrices based on these relationships, can be expressed as a set of action operators, a. Let Λ_{Ω} denote this reduced set of Λ-matrices based on the relationship reductions. Then the causaloid is denoted by the pair $\left(\Lambda_{\Omega}, a\right)$. Reductions by the third level of physical compression are manifested by identities that express higher order Λ-matrices in terms of lower order ones. Examples of Λ-matrix set reductions are in the following two scenarios:

(1) When the fiducial set for the composite region is separable (expressable) into (as) a cartesian product of the fiducial sets of the components, i.e.

$$\Lambda_{l_{x_1} l_{x_2} l_{x_3} \dots l_{x_n}}^{k_{x_1} k_{x_2} k_{x_3} \dots k_{x_n}} = \Lambda_{l_{x_1} l_{x_2} l_{x_3} \dots l_{x_m}}^{k_{x_1} k_{x_2} k_{x_3} \dots k_{x_m}} \Lambda_{l_{x(m+1)} l_{x(m+2)} \dots l_{x_n}}^{k_{x(m+1)} k_{x(m+2)} \dots k_{x_n}} \text{ when } \Omega_{x_1 \dots x_n} = \Omega_{x_1 \dots x_m} \times \Omega_{x_{(m+1)} \dots x_n} \tag{5.18}$$

(2) When higher order Λ-matrices can be computed based on pairwise 2-index Λ-matrices:

$$\Lambda_{l_{x_1} l_{x_2} l_{x_3} \dots l_{x_n}}^{k_{x_1} k_{x_2} k_{x_3} \dots k_{x_n}} = \sum_{k_2' \in \Omega_{23}, \dots, k_{n-1}' \in \Omega_{(n-1)n}} \Lambda_{l_1 k_2'}^{k_{x_1} k_{x_2}} \dots \Lambda_{l_{(n-1)} l_n}^{k_{(n-1)}' k_n} \text{ when } \begin{array}{l} \Omega_{12\dots n} = \Omega_{12} \times \dots \times \Omega_{(n-1)n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \Omega_{ij} = \Omega_i \times \Omega_j \end{array} \tag{5.19}$$

Next, consider the case where ensembles of experiments are limited or were one large data set card is instead collected. One considers this case because effects are not preserved as these repeated experiment processes are not invariably reversible, so that an experiment performed later would be run under very different conditions regardless of how hard one tries to preserve the cosmological laboratory. So, one considers running experiments in one long consecutive batch. However, this taxes the statistical theory behind any of the probability propositions arises from such an experiment. To overcome this, consider the following methodology. Let *A* be a proposition concerning the data that will be collected in an experiment. To this proposition associate a vector, r_A, as before with regions. Next, consider a complete sequence of mutually exclusive propositions, $\{A_i\}_{i=1,\dots,M}$, where $A_i = \left[A^C\right]^{C \dots C}$, is the i^{th} complementation of *A*.

Define the approximating vector, $r_A^I = r_A + \sum_{i=1}^{M} r_{A_i}$. Declare the assertion:

$$A \text{ has a}\left(\text{n}\right)\left(\text{approximately}\right) \text{ true value} \Leftrightarrow r_A \approx_\varepsilon r_A^I \tag{5.20}$$

where the equivalence $\approx_\varepsilon$ is modulo an approximation to within a threshold ε to be made precise later. This also points to the inevitability that experiments may never concisely estimate parameters. Now consider the vectors given by the application of the causaloid product $\otimes^\Lambda$:

$$r_n \equiv r_{(X_{1n},F_{1n})} \otimes^\Lambda r_{(X_{2n},F_{2n})} \tag{5.21}$$

where $R_n = R_{1n} \bigcup R_{2n},\ n = 1,...,N,\ N >> 1$. Next, define the vector:

$$r_n^I \equiv \sum_{Y_{1n} \subset F_1} r_{(X_{1n},F_{1n})} \otimes^\Lambda r_{(X_{2n},F_{2n})} \tag{5.22}$$

Now define the difference vector, $\overline{r}_n = r_n^I - r_n$ and assume the condition $r_n = pr_n^I,\ \forall n$. In this way, r_n plays the role of v and r_n^I that of u. To get to a calculation of the probability proposition, one now considers the vector definition:

$$r_A \equiv \sum_{(p-\Delta p)N<|S|<(p+\Delta p)N} \left(\bigotimes_{n\in S}^{\Lambda} r_n\right) \otimes^\Lambda \left(\bigotimes_{n\in \overline{S}}^{\Lambda} \overline{r}_n\right) \tag{5.23}$$

A translation of this vector is the following: r_A corresponds to the property that pN out of N regions R_n have the result X_{R_n} to within a threshold of $\pm\Delta pN$. One now has the condition, $r_A^I \equiv \bigotimes_n^\Lambda r_n^I$. Taking the definition $r_n = pr_n^I,\ \forall n$, and using an approximation to the binomial distribution, one can rewrite r_A:

$$r_A = \left[\sum_{(p-\Delta p)N<|S|<(p+\Delta p)N} p^n (1-p)^{N-n} \right] r_A^I \approx \left[1 - O\left(\frac{1}{\Delta p \sqrt{N}} \right) \right] r_A^I \qquad (5.24)$$

Hence, $\frac{r_A}{r_A^I} \approx 1 - O\left(\frac{1}{\Delta p \sqrt{N}} \right)$. For a given threshold $\varepsilon > 0$, $\left| \frac{r_A}{r_A^I} \right| < 1 - \varepsilon, \ \forall N > N_\varepsilon$ for a sufficiently large N_ε. In this respect, the equivalence, $\approx_\varepsilon$ occurs between r_A and r_A^I and the $\varepsilon -$truthfulness of *A*. The formal definition of a universal causaloid follows:

Definition: (Universal Causaloid). The universal causaloid for a region, *R*, made up of elementary regions $\{R_x\}$, when it exists is defined as the entity represented by a mathematical object which may be utilized to calculate the vectors, r_A for a proposition *A* concerning the data collected in $\{R_x\}$ such that if A is $\varepsilon -$truthfulness, one has that $r_A \approx_\varepsilon r_A^I$ where $r_A^I = r_A + \sum_{i=1}^{M} r_{A_i}$ and $\{A_i\}_{i=1,\ldots,M}$ is a complete set of mutually exclusive propositions where $A_i = \left[A^C \right]^{C\ldots^C}$, is the complementation *i* times of *A*.

By using the symmetries inherent in classical probability (CprobT) and quantum theory (QT), the calculation of the $\varepsilon -$truthfulness can be accomplished without repeated experiments within those paradigms. The universal causaloid is seen as corresponding to the entire history of the universe that is essential to calculations pertinent to cosmological constructs without the enormity of its computation through these more compact $\varepsilon -$truthfulness tests for propositions. An assumption that would further simplify the computations involved with universal causaloids is the principle of counterfactual indifference:

Definition. (Principle of Counterfactual Indifference). The principle of counterfactual indifference is the condition that the probability of an event *E* does not depend on the action that would have been implemented had the complement E^C happened instead if one conditions on cases where E^C did not happen modulo that the measurement device did not alter the state of the observed entity in any large way (low key measurement).

Applying this condition to the case of *r* vectors, $r_{(X_1,F_1)} = r_{(X_1,F_1^C)}$ where $X_1 \subseteq F_1, F_1^C$ since when one applies the procedure actions F_1 and F_1^C, one does the same thing when the same outcomes, X_1 are observed by counterfactual indifference. The universal causaloid is a macroscopic approach to physical theory construction. By combining this attribute with the promise of the discrete computational models of LQG at the planck scale, despite the unknown emergence of a 3+1 dimensional spacetime at that microscopic level, an emergent property of QG may be mended there. This is the proposal of this paper, utilizing a generalization to the probabilistic causaloid and the LQG-spinfoam inspired computation at the planck-scale pixels of a surface for describing abstract physically conformal information.

It has been shown that a version of a quantum (classical) computer can be setup using the causaloid formalism by considering an abstract computer with generalized gates that is a subset of all possible gates in a pseudo-lattice of pairwise interacting qubits. Call this pseudo-lattice of pairwise interacting qubits, Θ_L. Call the universe set of gates possible, S_I. In a practical computer, the set of gates is restricted to a finite number N.

Let $S \equiv \{s_i\}_{i=1}^{N} \subset S_I$ be the set of gates in a computer. Define a causaloid on the set of pairwise interacting qubits, Λ_S. The triple (Θ_L, Λ_S, S) is then considered a causaloid-induced computer on a pseudo-lattice of pairwise interacting qubits, Θ_L, with quantum gates *S*. Now consider the class of causaloid-induced computers with number of gates bounded above by *M*. Call this class, C_M^{Λ}. A universal computer in this class is one that can simulate all other computers in C_M^{Λ}. Here it should be pointed out that an important distinction between a QG computer and a quantum (classical) computer is that it is not a step computer, i.e., no sequential time steps are realized for computation. This is so because of the indefinite causal structure in a QG environment and subsequent computer.

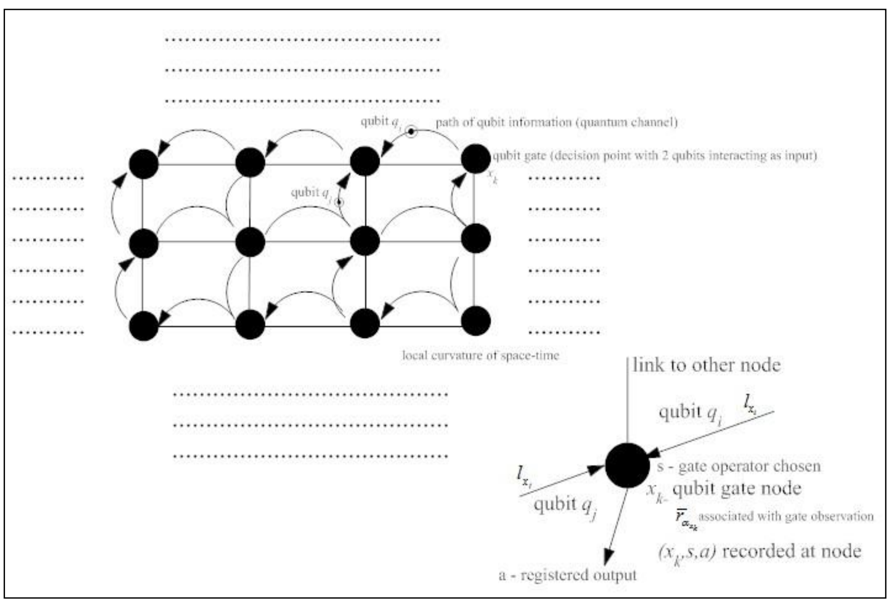

Figure 12 - QG Computer on Lattice

Each node of the pseudo-lattice represents a quantum gate, x_k, where a particular gate operator, *s* is chosen at interaction time between two input qubit information channels, q_i and q_j. Upon interaction and gate operation chosen, an output, *a*, is produced via measurement and transformation operators. The triple, (x_k, s, a) is recorded at the gate. Associated with this record is the vector, $\overline{r}_{\alpha_{x_k}}$. The two separate qubit channel inputs can then be separated as $l_{x_k} = l_{x_i} l_{x_j}$ where l_{x_i} and l_{x_j} mark the fiducial measurements on qubits q_i and q_i respectively. These operations and the pseudo-lattice constitute a causaloid diagram for a quantum computer. The causaloid for the pairwise interacting qubit computer model can be written as:

$$\Lambda = \left(\left\{ \Lambda_{\alpha_{x_k}}^{l_{x_i} l_{x_j}},\ \forall x_k \right\}, \left\{ \Lambda_{l_{x_i} l_{x_j}}^{k_{x_j} k_{x_w}},\ \forall \text{ adjacent } x_k, x_w \right\}; R \right) \tag{5.25}$$

where *R* is the set of rules (actions) constructing the causaloid qubit diagram (pairwise interacting qubits, nodes with gate operations as defined above) and the clumping operations given by the categories in (5.18) and (5.19) for grouping nonsequential nodes for any set in the set of all configurations of qubit nodes, Ω. State evolution can be simlulated by considering nested spacetime regions, $R_t,\ t = 0, \ldots, T$ where:

$$R = R_0 \supset R_1 \supset \ldots \supset R_T = \varnothing \tag{5.26}$$

Interprete the region R_t as what happens in *R* after time *t*. Now consider the state vector, $\overline{p}(t) = \overline{p}(R_t)$ at time *t* for the region R_t. Construct the evolution equation as:

$$\overline{p}(t+1) = G_{t,t+1}\left(\overline{p}(t)\right) \tag{5.27}$$

where $G_{t,t+1}$ is the evolution operator that depends on the output,-procedure pair, $\left(Y_{R_t \setminus R_{t+1}}, F_{R_t \setminus R_{t+1}}\right)$ on the complementary region, $R \setminus R_t$. By using this technique of nested regions, one simulates a time evolution without using a physical time parameter.

QG computers are conceivable and plausible if one can show that a GR computer is possible. Nonetheless, for the sake of completeness, a GR computer should be demonstratable using a causaloid formalism as have QT and classical computers above. Possible GR compatible computers may utilize gravitational waves and have been shown to be plausible Church-Turing-Deutsch physically-based computers leading to hypercomputability by utilizing supertasks (Pitowsky, 1990; Etesi & Nemeti, 2002; Shagrir & Pitowsky, 2003). Hypercomputability is the condition in a computing device that permits one to compute functions that cannot be computed by a Turing machine. These GR hypercomputers utilize a special spacetime structure called Malament-Hogarth spacetime.

Definition (Malamert-Hogarth spacetime). A pair $(\mathcal{M}, g)$, where $\mathcal{M}$ is a connected 4-dim Hausdorff C^{∞} manifold and g is a Lorentz metric, is called a *Malamert-Hogarth spacetime* if $\exists$ a timelike half-curve $\gamma_1 \subset \mathcal{M}$ and a point $p \in \mathcal{M} \ni \int_{\gamma_1} d\tau = \infty$ and $\gamma_1 \subset I^{-}(p)$ where $I^{-}(\gamma)$ denotes the set of past events of γ.

In an Malamert-Hogarth (M-H) spacetime $(\mathcal{M}, g)$ there is a future-directed timelike curve γ_2 that starts at a point q that is in the chronological past of p (i.e.,

$q \in I^{-}(p)$) and ends at p. So, $\int_{\gamma_2(q,p)} d\tau < \infty$. Furthermore, in an M-H spacetime, events are not related to each other causally, that is, an M-H spacetime is not globally hyperbolic and so, has an indefinite causal structure. Two other powerful classes of GR computers will be reviewed that are capable of computing general recursive functions and are more feasible cosmologically.

Definition. (past temporal string). Consider the string that is formed from a collection of nonintersecting open regions, $O_i \in (\mathcal{M}, g)$, an M-H spacetime, such that: (i) $\forall i,\ O_i \subset I^{-}(O_{i+1})$, and (ii) $\exists q \in \mathcal{M} \ni \forall i,\ O_i \subset I^{-}(q)$. Such strings are called past temporal strings (PTS).

PTSs construct complex spacetimes referred to as arithmetic-sentence-deciding spacetimes of order *n* or SAD_n. A first order *SAD*, denoted by SAD_1, is a Turing Machine (TM) that travels towards an event and is in the event's past spacetime cone. SAD_1s can be stacked on top of each of spacio-temporally to construct higher order SAD_n.

Result. A SAD_1 can decide 1-quantifier arithmetic, that is, any relation of the form $S(z) = \exists x R(x,z)$ or $\forall x R(x,z)$, where R is recursive.

Definition. If $(\mathcal{M}, g)$ is a M-H spacetime, then it is a SAD_1 spacetime. If $(\mathcal{M}, g)$ admits strings of SAD_{n-1} then it is a SAD_n spacetime.

SAD_n spacetimes construct hierarchies of spacetimes as in the following sequence:

$$FTM - TM - SAD_1 - .... - SAD_n - ... - AD \tag{5.28}$$

where an *AD* is an arithmetic-deciding computer which is a computer that can compute exactly $\aleph_0$ functions.

Now consider GR computers that can perform supertasks in the vicinity of back holes. Rotating black holes that are not charged are classified as Kerr black holes.If they are charged then they are called Kerr-Newman black holes. The exterior of black holes that are charged form a spacetime called a Kerr-Newman spacetime and are types of M-H spacetimes. Therefore, an abstraction for a GR computer utilizing the effects near a black hole is plausible. To this end, a scenario is built where two timelike curves, (γ_P, γ_O) are traced respectively, for a computer traveling around the black hole in a stable orbit and an observer crossing the outer event horizon of the black hole, entering the inner horizon, but not continuing into a singularity. Both computer and observer start from a point $q \in \mathcal{M}$ with $\|\gamma_P\| = \infty$ and $\|\gamma_O\| < \infty$.

The Malament-Hogarth event takes place at a point *p* on an orbit around the black hole. The role of the computing device is to decide on the consistency of theorems of ZFC and informing the observer of such results. Assume that a TM, labeled *T*, that is capable of enumerating all the theorems of ZFC exists and that the computing device *P* and observer *O* have a copy of it each. Then if the observer, *O*, does not receive a signal from *P*, before it reaches *p*, then the ZFC is consistent. Otherwise, if *P* receives a message before reaching *p*, then ZFC is inconsistent. This class of GR computers near black holes are referred to as relativisitc $G = (\gamma_O, \gamma_P)$ computers (Syropoulos, 2008, pp. 137-148).

A similiar, but ideologically different class of black hole relativisitc computers is that proposed by Lloyd and Ng. In this model, the entire black hole is considered as a simple but ultimate computer with speed v ops/bit/unit of time and with number of bits of storage memory, I, bounded from above according to (Ng & Lloyd, 2004):

$$Iv^2 \leq \frac{1}{t_p^2} \approx 10^{86}\,\sec^{-2} \tag{5.29}$$

Taking this to its physical conclusion, the entire universe is considered a self-referential, self-constructing computer and as such, any physical device or thing is a computer (Lloyd, 2000; Lloyd, 2006a). The seeds for a deterministic computing universe hypothesis were, of course, planted earlier by Zuse and others in the Zuse Thesis - the universe is a computer via deterministic cellular automaton (Schmidhuber, 1996; Zuse, 1970). More recently, Wolfram posits that if spacetime is discrete, cellular automaton model the universe and as such, are limited in their computation of things, but that everything is a computer of sorts (Wolfram, 2002)

Regardless, it is still unknown how time behaves at the planck scale, posited to be fuzzy, at best, in the QG research arena, not withstanding several controversial experiments minimizing or challenging this effect (Lieu & Hillman, 2003). Nonetheless, in a conceptual QG computer, the concept of separate space or time resources must be combined to reflect a new kind of singular spacetime resource measurement for showing computational rates and limitations. As pointed out before, QG computers are nonstep devices.

Appendix E: Category and Topos Theory

Consider an object C consisting of general objects, $A, B, C, \ldots$ labeled as $Ob(C)$ and maps or relations (sometimes referred more generally as arrows or morphisms), $f, g, h, \ldots$, labeled as $Arr(C)$, such that

1. for each arrow $f \in Arr(C)$, $\exists$ two objects, $dom(f), cod(f) \in Ob(C)$ such that f acts only on $dom(f)$ and maps only to $cod(f)$, i.e., $f: dom(f) \to cod(f)$, written as $dom(f) = A \xrightarrow{f} cod(f) = B$,
2. for each object $A \in Ob(C)$, an identity map, denoted by 1_A exists such that $A \xrightarrow{1_A} A$ is one such map from A to A,
3. for each pair of maps, (f, g) in $Arr(C)$ such that $A \xrightarrow{f} B \xrightarrow{g} C$, when objects $A, B,$ and C exists, a composition map $h = g \circ f$ exists, defined as $A \xrightarrow{h} C$,
4. if $A \xrightarrow{f} B$, then $1_B \circ f = f$ and $f \circ 1_A = f$ (identity laws), and
5. if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D,$ then $(h \circ g) \circ f = h \circ (g \circ f)$ (associative law).

Some consequences of this definition are (a) $dom(1_A) = cod(1_A) = A$, (b) $g \circ f$ is defined if $dom(g) = cod(f)$, (c) $dom(g \circ f) = dom(f)$, and (d) $dom(g \circ f) = cod(g)$. Label the pair of objects $Ob(C)$ and arrows $Arr(C)$ as C. If C satisfying only condition 1 it is called a *metagraph*. If in addition, C satisfies 2 then it is called a *metacategory*.

Metacategories will be subject to the axioms of 3 and 4. With some imagination, one can see the generality of metacategories. For example, the metacategory of sets consists of all sets and arrows are all functions with the usual identity and composition of functions defined in naïve set theory. The metacategory of all groups consists of all groups *G,H,K,…* with arrows which are functions *f* from a set *G* to a set *H* defined so that $f : G \to H$ is a homomorphisn of groups. The metacategory of all topological or compact Hausdorff spaces each with the continuous functions as arrows (topologies can be defined by continuous maps) are two other examples.

Definition. A category is a metacategory, $\mathcal{C}$, interpreted within set theory, that is, the objects in a category is a set of objects, *O* and the arrows is a set *A* of arrows, together with the usual functions defined by $\xrightarrow{dom}$ and $\xrightarrow{cod}$ such that $A \underset{cod}{\overset{dom}{\rightleftarrows}} O$.

Definition. The set of all possible arrows from the object *B* to *C* in $\mathcal{C}$, a category, is denoted as $\hom(B,C) = \{ f \mid f \text{ in } \mathcal{C}, dom(f) = B, cod(f) = C\}$, the set of its *morphisms*.

The set $\hom(A,A)$ defines all *endomaps*, for all objects *A* in $\mathcal{C}$, a category. A special type of category is the *monoid* which is a category with exactly one object. Indeed, a category is a very general animal which can be described as a generalized mathematical object reflecting the rich structure of specialized mathematical structures used in known diverse mathematical and scientific endeavors. In order to further develop the richness of categories, the definition of mapping between categories is given.

To generalize the ideas of a null set and singleton subsets we define initial and terminal objects of a category C .

Definitions. An object 0 is *initial* in a category C if for every $A \in Ob(C)$ there is one and only one arrow $f_A : 0 \to A$ in C . Reversing the role of arrows, an object 1 is *terminal* in C if for every $A \in Ob(C)$ there is one and only one arrow $f_A : A \to 1$ in C .

Duality is a mathematical concept in which the roles of two objects engaged in a structural relationship are reversed. In the general case of categories, which would generalize to dualities everywhere, one constructs:

Definition. From a given category C , construct its dual or opposite category, C^{op} in the following manner:

$Ob(C) = Ob(C^{op})$ and for each $f \in C$ mapping $A \to B$, define the arrow $f^{op} \in C^{op}$ mapping $B \to A$. The only arrows of C^{op} are of these constructions. The composition $f^{op} \circ g^{op}$ is defined precisely when $f \circ g$ is and $f^{op} \circ g^{op} = (g \circ f)^{op}$. In addition, $f^{op} = cod(f)$ and $cod(f^{op}) = dom(f)$.

The significance of duality in category theory is that if a statement Σ of category theory is held to be true then automatically the statement given by the opposite Σ^{op} is true as well. The conclusion is that this duality principle cuts in half the work to be done in a category or in category theory in general (Goldblatt, 2006). One would like to generalize the concept of products and limits since with these constructionists theories can be built. To this end define general diagrams and cones:

Definition. Let D be a metagraph (diagram) with vertices $\{d_i : i \in I\}$ for a category C. A *cone* over D is a family of arrows $\{A \xrightarrow{f_i} d_i : i \in I\}$ from A to objects in D such that for any arrow $d_i \xrightarrow{f_{ij}} d_j$ in D, the diagram

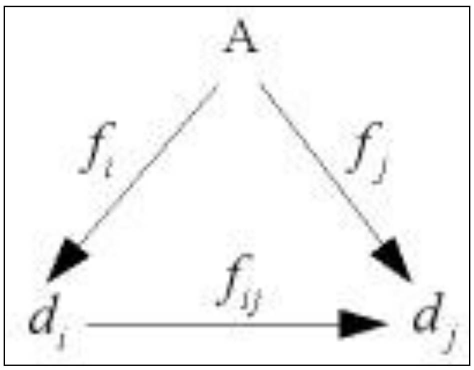

Figure 13 - Category cone

commutes. The object A is called the *vertex* of the cone. An arrow from a cone over D $\{A \xrightarrow{f_{A_i}} d_i : i \in I\}$ to another cone over D $\{B \xrightarrow{f_{B_i}} d_i : i \in I\}$ is a C*-arrow* $A \xrightarrow{g} B$ if the diagram

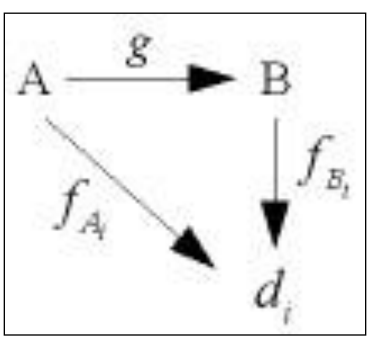

Figure 14 - Category arrow

commutes for each $i \in I$. If such an arrow *g* exists then the cone $\{A \xrightarrow{f_{A_i}} d_i : i \in I\}$ *factors through* the cone $\{B \xrightarrow{f_{B_i}} d_i : i \in I\}$. The set of cones over *D* denoted by *Cone(D)* then form a category using this procedure. One now gets to the definition of limits.

Definition. A *limit* for the diagram *D* is the terminal object of *Cone(D*). The *colimit* of *D* is the terminal object of the cone, $Cone^{opp}(D)$ which is the cone defined over the dual category, C^{op}.

Definition. A category, C is said to be *(finitely) complete* or *cocomplete* if the limit or colimit of any finite diagram in C exists in C.

A useful device for category manipulation is the pullback mechanism. Formally, a *pullback* of a pair of arrows defined as $A \xrightarrow{f} C \xleftarrow{g} B$ in $Arr(C)$ with common codomain, *C*, is a limit in C for the diagram:

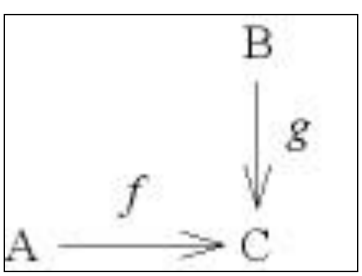

Figure 15 - Category pullback

where a cone for this diagram consists of a triplet of arrows (f', g', h) in C such that the diagram:

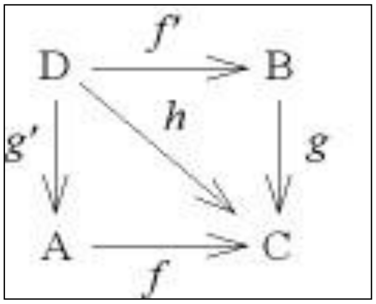

Figure 16 - Category pullback cone

commutes. Using the definition of a universal cone and the commutivity of the above diagram, one can eliminate the arrow *h* and arrive at a more precise definition,

Definition. A *pullback* of the pair of arrows $A \xrightarrow{f} C \xleftarrow{g} B$ in $Arr(C)$ is a pair of arrows $A \xrightarrow{g'} D \xleftarrow{f'} B$ in $Arr(C)$ such that:

(1) $f \circ g' = g \circ f'$ in $Arr(C)$, and

(2) whenever $A \xrightarrow{h} E \xleftarrow{j} B$ are a pair of arrows in $Arr(C)$ such that $f \circ h = g \circ j$ then there is exactly one arrow in $Arr(C)$ $k : E \to D$ such that $h = g' \circ k$ and $j = f' \circ k$. The diagram (f, g, f', g') is called a *pullback square* (Goldblatt, 2006).

Exponentiation is defined next. Consider the category given by the usual sets of axiomatic set theory with set operations. Denote this category by the label, *SET*. If *A* and *B* are two sets in *SET*, let $B^A = \{f : f : A \to B\}$ denote the set of all functions (arrows) having domain *A* and codomain *B*. A special arrow in *SET* will be associated with B^A, the evaluation arrow, $ev : B^A \times A \to B$ with the assignment rule, $ev((f, x)) = f(x)$.

Definition. A category $\mathcal{C}$ has *exponentiation* if (a) it has a limit for any two arrows in $Arr(\mathcal{C})$, and (b) if for any given objects $A, B \in Ob(\mathcal{C})$ there exist an object, B^A and an arrow, $ev \in Arr(\mathcal{C})$, $ev : B^A \times A \to B$, referred to as an evaluation arrow, such that for any $C \in Ob(\mathcal{C})$ and $g \in Arr(\mathcal{C})$, $g : C \times A \to B$, there exist a unique arrow, $\hat{g} \in Arr(\mathcal{C})$ making the diagram:

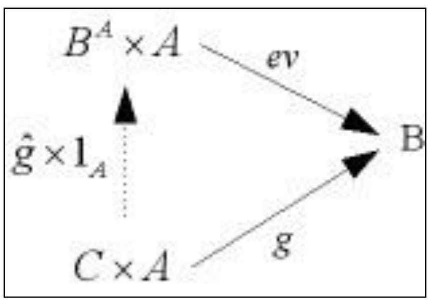

Figure 17 - Category exponentiation

commute, that is, the existence of a unique arrow, $\hat{g}$ such that $ev \circ (\hat{g} \times 1_A) = g$.

In order to compare two or more categories, a mechanism must exist that maps categories to each other. The space of morphisms between two categories will now be defined.

Definition. A *functor*, T is a morphism between two categories, $\mathcal{C}$ and $\mathcal{B}$, written as $T : \mathcal{C} \to \mathcal{B}$ in which $dom(T) = \mathcal{C}$ and $cod(T) = \mathcal{B}$, which assigns to each $C \in \mathcal{C}$, an object $T(C) \in \mathcal{B}$ and an arrow associated with T, written, T_{ar}, that assigns to each arrow

$f : C \to C^{'}$ of $\mathcal{C}$ an arrow $T_{ar} f : T(C) \to T(C^{'})$ of $\mathcal{B}$ in such a way so that

$T(1_C) = 1_{T(C)}$ and $T(g \circ f) = T(g) \circ T(f)$

whenever, $g \circ f$ is defined in *C*.

Functors on categories must act on both objects and arrows of categories as above. In this way a composition of functors, functor isomorphism, and a faithful functor can be defined to expand on the space of category functors and hence on the relations between categories.

Definitions. (a) A functor $S \circ T : \mathcal{C} \to \mathcal{A}$ is a *functor composition* of functors *S* and *T* if $\mathcal{C} \xrightarrow{T} \mathcal{B} \xrightarrow{S} \mathcal{A}$ are functors between categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ such that $C \to S(T(C))$ and $f \to S(T(f))$ for objects *C* and arrows *f* of $\mathcal{C}$. (b) A function $T : \mathcal{C} \to \mathcal{B}$ is a *functor isomorphism* between $\mathcal{C}$ and $\mathcal{B}$ if it is a bijection both on objects and arrows between $\mathcal{C}$ and $\mathcal{B}$, i.e., if $\exists$ a functor $S : \mathcal{B} \to \mathcal{C}$ for each functor *T*, such that $S \circ T = T \circ S = Id$ where *Id* is the identity functor between $\mathcal{C}$ and $\mathcal{B}$ and $S = T^{-1}$ is a two-sided inverse functor. (c) a functor $T : \mathcal{C} \to \mathcal{B}$ is *full* when to every pair $(C, C^{'})$ of objects in $\mathcal{C}$ and every arrow $g : T(C) \to T(C')$ of $\mathcal{B}$, $\exists$ an arrow $f : C \to C'$ in $\mathcal{C}$ with $g = T(f)$, and (d) a functor $T : \mathcal{C} \to \mathcal{B}$ is *faithful* (an embedding) if to every pair $(C, C^{'})$ of objects and every pair (f, g) of parallel arrows (arrows with the same domain and codomain) in $\mathcal{C}$, $T(f) = T(g) \Rightarrow f = g$. A consequence of these definitions is that compositions of faithful and full functors are again faithful and full respectively.

Faithfulness and fullness are embedding features between categories in the following sense: if (C, C') is a pair of objects in $\mathcal{C}$, the arrow of *T*, $T_{ar} : \mathcal{C} \to \mathcal{B}$ assigns to each $f : C \to C'$ an arrow $T_{ar}(f) : T(C) \to T(C')$ so that a function is defined:

$$T_{(C,C')} : \hom(C, C') \to \hom(T(C), T(C')), \quad f \to T(f)$$

as a mapping of the set of arrows between *C* and *C'* to the set of arrows between *T(C)* and *T(C')*, then *T* is full when every such function $T_{(C,C')}$ is surjective and faithful when it is injective. If *T* is both full and faithful, then every such $T_{(C,C')}$ is bijective, but not necessarily an isomorphism (MacLane, 1971). Embeddings of categories naturally call upon a definition of subcategories or categories contained within other categories.

Definitions. A *subcategory* $\mathcal{S}$ of a category $\mathcal{C}$ is a collection of objects and arrows of $\mathcal{C}$ that is closed under identities, domains and codomains, i.e., (a) if *f* is an arrow of $\mathcal{S}$ then it is an arrow of $\mathcal{C}$ and both $dom(f)$ and $cod(f)$ are objects of $\mathcal{S}$, (b) for each object *S* in $\mathcal{S}$, its identity arrow, 1_S is in $\mathcal{S}$, and (c) for every pair of arrows (f, g) in $\mathcal{S}$,their composition, $g \circ f$ is in $\mathcal{S}$. Consequently, $\mathcal{S}$ is also a category. An injection map, $T_{\mathcal{S}\,inj} : \mathcal{S} \to \mathcal{C}$ sending objects and arrows of $\mathcal{S}$ to itself in $\mathcal{C}$ is called the *inclusion functor*. It is consequently faithful. $\mathcal{S}$ is called a *full subcategory* of $\mathcal{C}$ when $T_{\mathcal{S}\,inj}$ is full (MacLane, 1971).

A useful example of a functor which will play an important role in mapping structures in a set-theoretic setting to a category-theoretic setting is the so-called forgetful functor denoted as $FOR: C \to SET$ where *SET* is the category of ordinary sets in set theory and *C* is any mathematical system (category). FOR strips off the extra structure attached to *C* and produces just the set objects of C as a new simply set category. FOR essentially forgets any structure (arrow rules, etc.) that *C* may have had, i.e., for a category C , if $A \in Ob(C)$, $FOR(A) = S_A$ where S_A is the strict set part of *A* and $FOR(f) = f$ for any $f \in Arr(C)$ .Versions of partially forgetful functors have been presented such as the class introduced by Geroch (1985) in which a functor is partially forgetful if it strips a category down to the categorically nearest simpler category (i.e., nearest meaning being in the same category, but with less structure morphologically). These lesser structured categories may then be mapped back to the richer categories that were stripped down by free construction functors that would then reintroduce the original richer structure back.

As an example, the Abelian group category, ABLGRP, can be stripped back entirely by FOR, but if we introduce a partially forgetful functor that just strips away commutivity, ANTICOM the group category, GRP is produced. By applying a free construction functor, COM that reintroduces commutivity back into GRP, one obtains ABLGRP. This will serve in producing a categorical chain of categories in which the repeated application of partially forgetful functors in combination with free construction functors will produce a family of categorically related structures and hence, a metachain for model-theories and their mathematical structures.

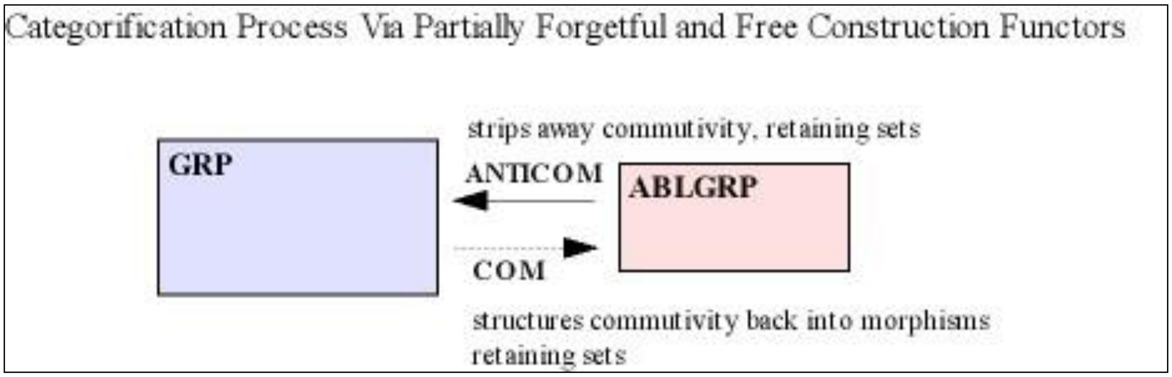

Figure 18 - Categorification process

A general way of defining a natural transformation of one functor to another in such a way that commutes between categories is through the following:

Definition. For two functors, $S, T : \mathcal{C} \to \mathcal{B}$, a *natural transformation* that maps *S* to *T*, denoted by $\tau : S \to T$, is a function that assigns to an object *C* of $\mathcal{C}$, an arrow $\tau_C : S(C) \to T(C)$ of $\mathcal{B}$ such that every arrow $f : C \to C'$ in $\mathcal{C}$ commutes in the following diagram:

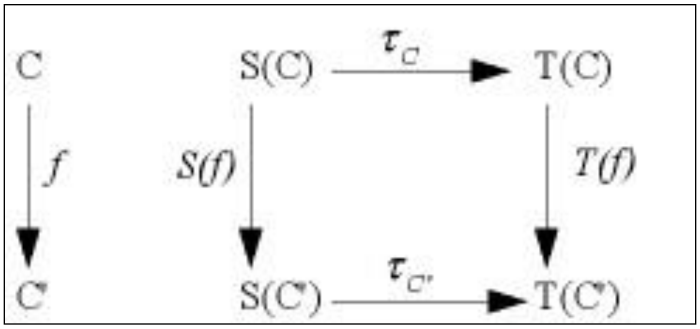

Figure 19 - Category natural transformation

The transformation, τ_C is called *natural in C* (MacLane, 1971). The notion of generalized categorical subsets, known as subobjects and the mechanism to find subobjects, a subobject classifier, will be discussed next.

Definition. An arrow $f \in Arr(C)$, $f : A \to B$ is called *monic* if for any parallel pair of arrows, $g, h : C \to A$ in $Arr(C)$ $f \circ g = f \circ h \Rightarrow g = h$.

Definitions. A *subobject* of an object, $D \in Ob(C)$ is a monic arrow in $Arr(C)$, $f : A \to D$, with codomain *D*. The set of all such subsets of *D* (if *D* is an ordinary set) is called the *powerset* of *D*, denoted by $\mathcal{P}(D)$ or 2^D.

Ordinary set inclusion, $\subseteq$ defines a partial ordering in $\mathcal{P}(D)$ so that $(\mathcal{P}(D), \subseteq)$ becomes a poset and hence a category in which the role of arrows is $A \to B \Leftrightarrow A \subseteq B$. Inclusion arrows then become commutative, reflexive, and transitive between subobjects. A generalization to 2^D in any category is the set of power objects denoted as Ω^D where the universe of discourse generalizes the binary set $\{0,1\}$.

Definition. A category, $\mathcal{C}$ with limits is said to have *power objects* if to each object, $A \in Ob(\mathcal{C})$, there are objects $\mathcal{P}(A)$ and $\in_A$, and a monic arrow $\in\colon \in_A \rightarrow \mathcal{P}(A) \times A$, such that for any object $B \in Ob(\mathcal{C})$ and relation map given by $r : R \rightarrow B \times A$, there is exactly one arrow $f_r : B \rightarrow \mathcal{P}(\mathcal{C})$ for which there is a pullback in $\mathcal{C}$ taking on the form,

$$\begin{tikzcd} R \arrow[r, "r", "\text{monic}"'] \arrow[d] & B \times A \arrow[d, "f_r \times 1_A"] \\ \in_A \arrow[r, "\in", "\text{monic}"'] & \mathcal{P}(\mathcal{C}) \times A \end{tikzcd}$$

Figure 20 - Category power object pullback

A *relation map* is a map with domain consisting of a relation *R*, which is an object such that $R \subseteq A \times B$ in which $(x, y) \in R \Leftrightarrow y \in f_R(x)$ where $f_R : A \rightarrow B$ is an arrow appropriately defined for the inclusion in *R*.

Definition. In a category, $\mathcal{C}$ with terminal object 1, a *subobject classifier* for $\mathcal{C}$ is an object $\Omega \in Ob(\mathcal{C})$ together with an arrow, $true : 1 \rightarrow \Omega$ that satisfies the following axiom:

Ω-axiom. For each monic arrow, $f : A \to D$, there is one and only one arrow, $\chi_f : D \to \Omega$ such that the diagram:

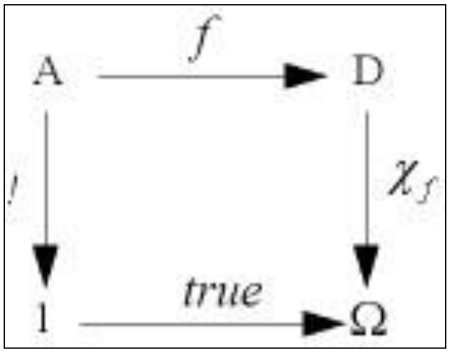

Figure 21 - Category subobject classifier pullback square

is a pullback square. Here χ_f is called the *character* of the monic arrow *f* (as a subobject of *D*), *true* is the arrow assigning a truth value of TRUE from the universe of discourse of truths, Ω, and *!* is the composition arrow defined by $true^{-1} \circ \chi_f \circ f : A \to 1$. The arrow $true^{-1}$ simply maps the value TRUE in Ω to the terminal object 1 in $Ob(C)$.

Enough structure has been defined to develop the formal definition of a Topos, which will serve as the template for a generalization to physical logic systems employed by information fields as defined in this dissertation.

Definition. An *elementary topos* is a category, C such that

(1) C is finitely complete,

(2) C has exponentiation, and

(3) C has a subobject classifier (Lawvere & Schanuel, 1997).

Alternatively, a category C is a *topos* if

(1) C is finitely complete, and

(2) C has power objects (Wraith, 1975).

For purposes of defining categories for recursive games (as product spaces of games), we consider category products. Products are considered as special cases of limits as has been reviewed earlier. Here we explicitly define products through morphisms (arrows) that are product projections to each component space. Formally, let I be a finite (discrete) index category. Define a product as follows:

Def. A category X is the product of a series of categories $\{X_i\}_{i\in I}$ if and only if there exists morphisms (canonical projections) $\pi_i : X_i \to X$ such that for every Y and family of morphisms $f_i : Y \to X_i$, there exist a unique morphism $f : Y \to X$ such that the diagram below commutes for every $i \in I$.

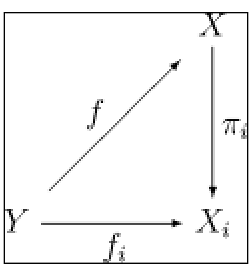

Figure 22 - Category product using canonical projection morphisms

This product category is denoted as $X = \prod_{i\in I} X_i$ with projection morphisms $(f_i)_{i\in I}$. For countable products, a product is viewed as a special categorical limit using the category index space I_n mapped to the finite product $\prod_{i\in I_n} X_i$, by a functor, $\nu_n : I_n \to \prod_{i\in I_n} X_i$. If the set of index categories are sequentially inclusive and approach a countable set, *I*, $I_i \subseteq I_{i+1}...$, and $\lim_{n\to\infty} I_n \equiv I$, then using the category definition of a limit and the cone $\left(I,\{f_i\}_{i\in I}\right)$, one concludes that the cone $\left(I,\{\pi_i\}_{i\in I}\right)$, using the projections $\{\pi_i\}_{i\in I}$, is the product limit.

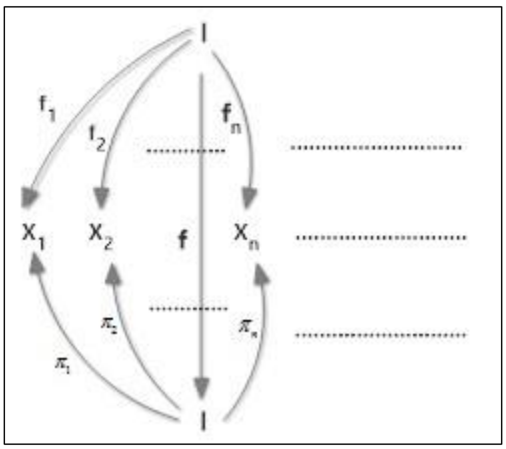

Figure 23 - Category product using category cone limit

The concept of *n*-categories and a well defined generalization to (∞, n)-category theory will be briefly discussed. The category theory outlined so far is of the 1-category where objects and morphisms mapping objects to other objects structure a category. The 0-category is simply a point set. The strict 2-category structure consists of objects, morphisms, and morphisms of morphisms called 2-morphisms. In general, the strict *n*-category is iteratively defined as repeated morphisms of morphisms, *n* times or *n*-morphisms. This *n*-category is sometimes written as *n*Cat. In the case of 2-categories, the more general class of bicategories needs more definition of operations in order to be well defined. Instead of equational equality, morphisms are equivalent up to isomorphisms. The Stasheff pentagon identity and MacLane's coherence theorem suffice to show a well defined diagram for bicategorical equivalence (Baez, 1997). The diagram below must commute in a bicategory,

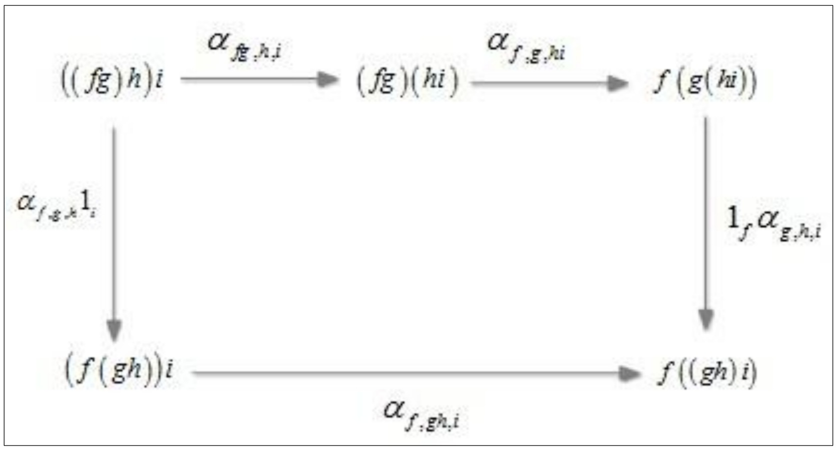

Figure 24 - Stasheff pentagon diagram

for morphisms *f, g, h*, and *i*, 1_i and 1_f are identity morphisms, and the α's are the appropriate 2-morphisms. Additionally, the following diagram must also commute,

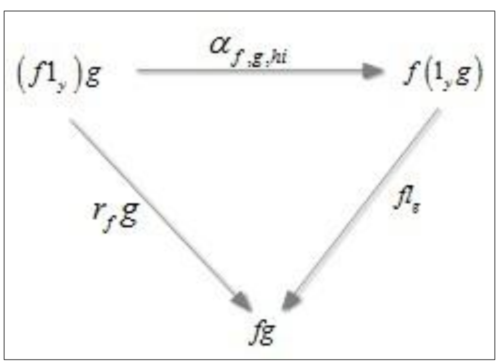

Figure 25- Triangle commute diagram for bicategory

given the morphisms, $f : x \to y, g : y \to z$, and the 2-morphisms that are the left and right identity constraints respectively, $l_f : 1_x f \Rightarrow f$, $r_f : f1_y \Rightarrow f$, for every morphism

$f: x \to y$. Lastly, the associators and unit constraints are natural transformations, and vertical composition of 2-morphisms is associative, vertical and horizontal composition of 2-morphisms are interchangeable and 1_x are identifies for vertical composition of 2-morphisms. This process of producing a bicategory from a strict 2-category is called *weakening* and produces a series of weak versions of *n*-categories known as *weak n*-categories. The idea is that the *weak n*-categories are more interesting because they have direct applications in physical structures.

In the higher *n*-categories, the *j*-morphisms are visualized as *j*-dimensional solids with boundaries defining source and targets of those *j*-morphisms. Strict *n*-categories use *j*-morphism representations called globes as the building blocks for defining *(j+1)*-morphisms and their well-defined operations. *Weak n*-categories approach these representations where equality of morphisms is at the top level *n* and isomorphisms define equivalences for lower level morphisms below that. Essentially, to map equivalences in *n*-categories, must look at the *(n+1)*-categories. See Leinster (2001) for a diversity of definitions and approaches to *n*-categories. Taking the trivial limit of *weak n*-categories, the ∞-category (or ω-category) is the category of all weak categories.

The concept of (n,r)-categories are categories that are *enriched* by the condition that all *j*-morphisms are equivalent (and hence reversible) for $j > r$, and any two parallel *j*-morphisms are equivalent for $j > n$. This still works to define (∞, n)-categories. The fullest (richest) such category is the $(\infty,1)$-category and all such enriched categories can be studied from that one (*n*Lab, 2012).